IMPORTANT THEOREMS AND DEFINITIONS

1. Limits and Continuity

Definition 1. *Let f be defined on an open interval around a but not necesarily at the point a itself. We write $\lim_{x\to a} f(x) = L$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that if x satisfies $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.*

Definition 2. *We write $\lim_{x\to a^+} f(x) = L$, the right hand limit of $f(x)$ as x approaches a, if for all $\epsilon > 0$ there is a $\delta > 0$ such that if x satisfies $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$. We write $\lim_{x\to a^-} f(x) = L$, the left hand limit of $f(x)$ as x approaches a, if for all $\epsilon > 0$ there is a $\delta > 0$ such that if x satisfies $0 < a - x < \delta$, then $|f(x) - L| < \epsilon$.*

Definition 3. *We write $\lim_{x\to a} f(x) = \infty$, if for all $M > 0$ there is a $\delta > 0$ such that if x satisfies $0 < |x-a| < \delta$, then $f(x) > M$. We write $\lim_{x\to a} f(x) = -\infty$, if for all $M < 0$ there is a $\delta > 0$ such that if x satisfies $0 < |x-a| < \delta$, then $f(x) < M$.*

Definition 4. *We write $\lim_{x\to\infty} f(x) = L$ if for all $\epsilon > 0$ there is an $M > 0$ such that if $x > M$, $|f(x) - L| < \epsilon$. We write $\lim_{x\to-\infty} f(x) = L$ if for all $\epsilon > 0$ there exists an $M < 0$ such that if $x < M$, then $|f(x) - L| < \epsilon$.*

Definition 5. *We write $\lim_{x\to\infty} f(x) = \infty(-\infty)$ if for all $M > 0$ $(m < 0)$ there is an $N > 0$ such that if $x < N$, then $f(x) > M$ $(f(x) < m)$. We write $\lim_{x\to-\infty} f(x) = \infty(-\infty)$ if for all $M > 0$ $(m < 0)$ there is an $N < 0$ such that if $x < N$, then $f(x) > M$ $(f(x) < m)$.*

Theorem 1. *If $\lim_{x\to a} f(x) = L$ and $lim_{x\to a} g(x) = M$ ($a = \pm\infty$ also works), and c is any constant,*

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} cf(x) = c(\lim_{x\to a} f(x))$ *(Constant Multiple Law)*
3. $\lim_{x\to a} f(x) \pm g(x) = L \pm M$ *(Sum and Diference Laws)*
4. $\lim_{x\to a} f(x)g(x) = LM$ *(Product Law)*
5. *If* $M \neq 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ *(Quotient Law)*
6. $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$, *if* $L > 0$

Theorem 2. *Let $r > 0$ be any real number. Then*

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If x^r is defined on $(-\infty, 0)$, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Theorem 3 (Squeeze Theorem)**.** *Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in an interval around a. Then if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Definition 6. *A function f is said to be* **continuous at** *a if $\lim_{x\to a} f(x) = f(a)$. There are three implications of this definition:*

(1) *$f(a)$ is defined.*
(2) *$\lim_{x\to a} f(x)$ exists.*
(3) *$\lim_{x\to a} f(x) = f(a)$*

Definition 7. *A function is continuous on an open interval if it is continuous at every point in the open interval. A function is said to be* **continuous from the left** *or* **continuous from the right** *of a point a if $\lim_{x\to a^-} f(x) = f(a)$ or $\lim_{x\to a^+} f(x) = f(a)$, respectively. A function is said to be continuous on an interval I if it is continuous on every open interval contained in I and is continuous from the left or right of any right or left hand endpoints that are contained in I. A function is continuous on its domain if it is continuous on every point in its domain.*

Lemma 1. *If f and g are continous at a point a and c is any constant, then the following are continuous at a:*

(1) cf
(2) $f + g$
(3) $f - g$
(4) fg
(5) $\frac{f}{g}$ *if* $g(a) \neq 0$

Theorem 4. *The following functions are known to be continuous at every point in their domain.*

(1) *Polynomials.*
(2) *Rational functions.*
(3) *Root functions.*
(4) *Trigonometic functions.*
(5) *Logarithmic and Exponential functions.*

Theorem 5 (The Intermediate Value Theorem)**.** *Suppose that f is continuous on a closed interval $[a, b]$ and that N is some number strictly between $f(a)$ and $f(b)$, then there exists a number c in (a, b) such that $f(c) = N$.*

Theorem 6. *Three limits involving trigonometric functions commonly appear. They are:*

(1) $\lim_{x\to 0} \frac{sin(x)}{x} = 1$
(2) $\lim_{x\to 0} \frac{1-cos(x)}{x} = 0$
(3) $\lim_{x\to 0} sin(\frac{1}{x})$ *DNE*

Definition 8. *A function f is said to have a* **vertical asymptote** *at $x = a$ if any one of the following four conditions occurs:*

(1) $\lim_{x\to a^+} f(x) = \pm\infty$
(2) $\lim_{x\to a^-} f(x) = \pm\infty$

Definition 9. *A function is said to have a* **horizontal asymptote** *at $y = L$ if either of the following two conditions occurs. It is therefore possible to have more than one horizontal asymptote.*

(1) $\lim_{x\to\infty} f(x) = L$
(2) $\lim_{x\to-\infty} f(x) = L$

2. Derivatives

Definition 10. *The* **derivative** *of f at the point a, denoted $f'(a)$, is defined to be*

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists. Equivalently,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

A function f is said to be differentiable at a if the above limits exist at a. The **derivative function** *is the function whose domain is the set of points where f is differentiable and whose value at a point x is $f'(x)$, the derivative of f at x. If $y = f(x)$, the derivative function is often denoted by y' and by $\frac{d}{dx}y = \frac{dy}{dx} = f'(x)$*

Definition 11. *The* **tangent line** *to a function f at a point a is the line through the point $(a, f(a))$ with slope $f'(a)$. The equation of this line is $y = f(a) + f'(a)(x - a)$.*

Theorem 7. *If f is differentiable at a, then f is continuous at a. The converse of this statement is not true.*

Theorem 8. *Suppose that f and g are differentiable at a point x and c is a constant. Then the following are differentiable at x with the derivative given.*

(1) $\frac{d}{dx}c = 0$
(2) $\frac{d}{dx}cf(x) = cf'(x)$
(3) $\frac{d}{dx}f(x) \pm g(x) = f'(x) \pm g'(x)$ *(Sum and Difference Rules)*
(4) $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x)$ *(Product Rule)*
(5) $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}$, *as long as $g(x) \neq 0$ (Quotient Rule)*

Theorem 9. *The following functions are known to be differentiable at all points in their domain with the derivative given:*

(1) $\frac{d}{dx}x^n = nx^{n-1}$ *for all real n, for x where x^n is defined.*
(2) $\frac{d}{dx}sin(x) = cos(x)$
(3) $\frac{d}{dx}sec(x) = sec(x)tan(x)$

(4) $\frac{d}{dx}tan(x) = sec^2(x)$
(5) $\frac{d}{dx}cos(x) = -sin(x)$
(6) $\frac{d}{dx}csc(x) = -csc(x)cot(x)$
(7) $\frac{d}{dx}cot(x) = -csc^2(x)$

Theorem 10 (Chain Rule)**.** *If $u = g(x)$ is differentiable at x and $y = f(u)$ is differentiable at $u = g(x)$, then $(f \circ g)(x)$ is differentiable at x and the derivative is:*

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

or equivalently:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

There are three interpretations of the derivative of a differentiable function f at a point a:

(1) The derivative at a is the slope of the tangent line at a.
(2) The derivative is the limit of the slopes of the secant lines through a.
(3) The derivative is the instantaneous rate of change of function with respect to it's variable. Similarly, the derivative is the limit of the average rates of change.

Definition 12. *The* **linearization** *of a differentiable function f is defined to be*

$$P_1(x) = f(a) + f'(a)(x - a)$$

The linearization is the best linear approximation to a a differentiable function at a point. The best quadratic approximation to a function f at a point a is given by

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

The **Taylor Remainder** *is defined to be $R_i(x) = f(x) - P_i(x)$, for i=1,2.*

Theorem 11 (Taylor's Remainder Theorem)**.** *If $|f''(x)| \leq M$ for all x satisfying $|x - a| < \delta$, then*

$$|R_1(x)| \leq \frac{M}{2}(x - a)^2$$

for all x such that $|x - a| < \delta$.
If $|f'''(x)| \leq M$ for all x satisfying $|x - a| < \delta$, then

$$|R_2(x)| \leq \frac{M}{6}|x - a|^3$$

for all x such that $|x - a| < \delta$.

Definition 13. *Let $y = f(x)$ where f is differentiable. We define a new independant variable dx called the differential. Associated to this independant variable is the dependant variable dy, also called a differential. The differential dy is defined by the equation $dy = f'(x)dx$.*

3. Applications of the Derivative

Definition 14. *Let f be a function. f is said to have an* **absolute maximum** *at c if $f(x) \leq f(c)$ for all x in the domain of f. The number $f(c)$ is called the maximum value of f. f is said to have an* **absolute minimum** *at c if $f(c) \leq f(x)$ for all x in the domain of f. The number $f(c)$ is called the minimum value of f. The word "global" is often used instead of "absolute".*

Definition 15. *Let f be a function. f is said to have a* **local maximum** *at c if $f(x) \leq f(c)$ for all x in some open interval I containing c, where I must be contained in the domain of f. Let f be a function. f is said to have a* **local minimum** *at c if $f(c) \leq f(x)$ for all x in some open interval I containing c, where I must be contained in the domain of f. The word "relative" is often used instead of "local".*

Theorem 12 (The Extreme Value Theorem)**.** *Let f be a continuous function defined on a closed interval $[a, b]$. Then f achieves an absolute maximum and an absolute minimum somewhere in $[a, b]$. In other words, there exists a c and d such that $f(x) \leq f(c)$ for all x in $[a, b]$ and $f(d) \leq f(x)$ for all x in $[a, b]$.*

Theorem 13 (Fermat's Theorem)**.** *If f has a local maximum or a local minimum at a point c and $f'(c)$ exists, then $f'(c) = 0$.*

Definition 16. *A* **critical number** *of a function f is a point c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.*

Theorem 14 (The Closed Interval Method)**.** *Let f be a continuous function on a closed interval $[a, b]$.*

(1) *Find the values of f at the critical numbers of f.*
(2) *Find the values of f and the endpoints a and b.*
(3) *The largest of the values from (1) and (2) is the absolute maximum value. The smallest of the values from (1) and (2) is the absolute minimum value.*

Theorem 15 (Rolle's Theorem)**.** *Let f be a function defined on $[a, b]$ that satisfies the following three conditions.*

(1) *f is continuous on $[a, b]$*
(2) *f is differentiable on (a, b)*
(3) *$f(a) = f(b)$*

Then there is a c in (a, b) such that $f'(c) = 0$.

Theorem 16 (The Mean Value Theorem)**.** *Let f be a function defined on $[a, b]$ that satisfies the following two conditions.*

(1) *f is continuous on $[a, b]$.*
(2) *f is differentiable on (a, b).*

Then there is a c in $[a, b]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Definition 17. *A function f is said to be* **increasing** *on an interval I if $x_1 < x_2$ implies $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I. A function f is said to be* **decreasing** *on an interaval I if for all $x_1 < x_2$ implies $f(x_2) < f(x_1)$ for all x_1, x_2 in I.*

Theorem 17 (Increasing/Decreasing Test)**.** *If $f'(x) > 0$ for all x in an interval then f is increasing on I. If $f'(x) < 0$ for all x in I, then f is decreasing on I.*

Theorem 18 (The First Derivative Test)**.** *Let f be continous on an open interval I and let c be a critical point of f.*

(1) *If f' is positive to the near left of c and negative to the near right of c, then f has a local maximum at c.*
(2) *If f' is negative to the near left of c and postive to the near right of c, then f has a local minimum at c.*
(3) *If f' does not change sign at c, then f has neither a local maximum nor a local minimum at c.*

Definition 18. *A function is said to be concave up on an interval I if for all x and y in I, the secant line connecting the points $(x, f(x))$ and $(y, f(y))$ lies on or above the graph of f. A function is said to be concave down on an interval I if for all x and y in I, the secant line connecting the points $(x, f(x))$ and $(y, f(y))$ lies on or below the graph of f. Linear functions are both concave up and concave down.*

Theorem 19 (Concavity Test)**.** *If $f''(x) > 0$ for all x on an interval I then f is concave up on I. If $f''(x) < 0$ for all x on an interval I then f is concave down on I.*

Definition 19. *A function f is said to have an* **inflection point** *at a point c in its domain if the concavity of f is opposite to the near left of c than it is to the near right of c.*

Theorem 20 (The Second Derivative Test)**.** *Suppose that f'' exists and is continuous near c.*

(1) *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*
(2) *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*

Theorem 21 (The First Derivative Test for Absolute Extrema)**.** *Suppose that c is a critical number of a continuous function f defined on an interval.*

(1) *If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has an absolute maximum at c.*
(2) *If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has an absolute minimum at c.*

Definition 20. *A function F is said to be an* **antiderivative** *of f on an interval I if $F'(x) = f(x)$ for all x in I.*

Definition 21. *The* **indefintite integral** *of a function f on an interval I is the class of antiderivatives of f on I i.e. the class of functions whose derivative is f on I. This class is denoted by the symbols*

$$\int f(x)\,dx$$

The function f is called the **integrand** *and the symbol $\int$ is called the* **integral sign**.

Theorem 22. *If $f'(x) = 0$ for all x in an interval I, then f is a constant on I.*

Theorem 23. *If $f'(x) = g'(x)$ for all x is an interval I, then $f - g$ is a constant on I.*

Theorem 24. *Let F be any antiderivative of f on an interval I. Then the indefinite integral of f on I is the set of translations of F by a constant. Therefore we write:*

$$\int f(x)\,dx = F(x) + C$$

Theorem 25 (Properties of the Indefinite Integral)**.** *Let f and g be functions and a a constant. Then*

(1) $\int a\,dx = ax + C$
(2) $\int cf(x)\,dx = c\int f(x)\,dx$
(3) $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Some common antiderivatives are:

$\int x^n dx = \frac{1}{n+1}x^{n+1} + C; x \neq -1$	$\int cf(x) \pm g(x)dx = c\int f(x)dx \pm \int g(x)dx$
$\int sin(x)dx = -cos(x) + C$	$\int cos(x)dx = sin(x) + C$
$\int sec(x)tan(x)dx = sec(x) + C$	$\int csc(x)cot(x)dx = -csc(x) + C$
$\int csc^2(x)dx = -cot(x) + C$	$\int sec^2(x)dx = tan(x) + C$

4. Area and the Definite Integral

Definition 22. *Let $f \geq 0$ be a function. Let P be any partition of $[a, b]$,*

$$P = \{a_0 = a < a_1 < a_2 < \ldots < a_{n-1} < a_n = b\}$$

Let $\Delta x_k = a_k - a_{k-1}$ for $1 \leq k \leq n$. Define $||P|| = max\{\Delta x_k\}$. Choose points x_k^* in the intervals $[a_{k-1}, a_k]$. Then the area below the function and above the x axis is

$$A = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

if the limit exists.

Definition 23. *Let f be a function defined on a closed interval $[a, b]$. Let P be any partition of $[a, b]$,*

$$P = \{a_0 = a < a_1 < a_2 < \ldots < a_{n-1} < a_n = b\}$$

Let $\Delta x_k = a_k - a_{k-1}$ for $1 \leq k \leq n$. Define $||P|| = max\{\Delta x_k\}$. Choose points x_k^* in the intervals $[a_{k-1}, a_k]$. Then the **definite integral** *of f from a to b is*

$$\int_a^b f(x)\,dx = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

if the limit exists. If the limit does exist, f is said to be **integrable**. *$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ is called a* **Riemann sum**. *a is called the* **lower limit** *of integration and b is called the* **upper limit** *of integration. The function f is referred to as the* **integrand** *and the symbol $\int$ is referred to as the* **integral sign**.

Definition 24. *Some common choices for the points x_k^* are:*

(1) *Left endpoint method: $x_k* = a_{k-1}$ for $1 \leq k \leq n$.*
(2) *Right endpoint method: $x_k^* = a_k$ for $1 \leq k \leq n$.*
(3) *Midpoint method: $x_k^* = \frac{a_k + a_{k-1}}{2}$ for $1 \leq k \leq n$.*
(4) *Maximum value method: x_k^* = point where f achieves its largest value on $[a_{k-1}, a_k]$ for $1 \leq k \leq n$.*
(5) *Minimum vlaue method: x_k^* = point where f achieves its smallest value on $[a_{k-1}, a_k]$ for $1 \leq k \leq n$.*

Definition 25. *A partition is said to be* **regular** *if $\Delta x_k = \Delta x_j$ for all k and j.*

Theorem 26. *If f is integrable on $[a, b]$ then $\int_a^b f(x)\,dx$ may be found using a sequence of regular partitions and any choice of x_k^*. For example, if f is integrable we can find $\int_a^b f(x)\,dx$ by using the right endpoint method with $\Delta x = \frac{b-a}{n}$. In this case,*

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right)\frac{b-a}{n}$$

Theorem 27. *If f is continuous on $[a, b]$, increasing on $[a, b]$, or decreasing on $[a, b]$, then f is integrable on $[a, b]$.*

Theorem 28 (Properties of the Definite Integral)**.** *If all of the following integrals exist, then*

(1) $\int_a^a f(x)\,dx = 0$.
(2) $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

(3) *For any real number c,* $\int_a^b c\,dx = c(b-a)$
(4) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)dx$.
(5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Theorem 29 (Order Properties of the Definite Integral)**.** *If all the definite integrals below exist and $a \le b$, then*

(1) *If $f \ge 0$ then* $\int_a^b f(x)\,dx \ge 0$.
(2) *If $f(x) \ge g(x)$ for all x in $[a, b]$ then* $\int_a^b f(x)\,dx \ge \int_a^b g(x)dx$.
(3) *If $m \le f(x) \le M$ for all x in $[a, b]$, then*

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

(4) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Theorem 30 (The Fundamental Theorem of Calculus)**.** *Let f be a contiuous function defined on a closed interval $[a, b]$.*

(1) *Define $g(x) = \int_a^x f(t)\,dt$. Then $g'(x) = f(x)$.*
(2) *If F is any antiderivative of f, then*

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Theorem 31 (The Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))\,dx = \int f(u)du$$

Theorem 32 (The Substitution Rule for Definite Integrals)**.** *If g' is continuous on $[a, b]$ and f is continuous on the range of g, then*

$$\int_a^b f(g(x))\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Definition 26. *The area between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is*

$$A = \int_a^b |f(x) - g(x)|\,dx$$

5. Vectors and Vector Analysis

Definition 27. *A* **2 − dimensional vector***, denoted $\vec{v}$, is an ordered pair of real numbers $< v_1, v_2 >$. A* **representation** *of the vector $\vec{v} =< v_1, v_2 >$ is a directed line segment in the plane from a point $A(x, y)$ to $B(x+v_1, y+v_2)$. The* **position vector** *of a point $P(x, y)$ is $< x, y >$ with chosen representation the directed line segment from the origin to $P(x, y)$.*

Theorem 33. *Given a directed line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$, we may recover the vector $\vec{v}$ which it represents via the following formula:*

$$\vec{v} =< x_2 - x_1, y_2 - y_1 >$$

Definition 28. *The* **length** *of a vector $\vec{v} =< v_1, v_2 >$ is defined to be*

$$\| \vec{v} \| = \sqrt{v_1^2 + v_2^2}$$

Definition 29 (Definition of Vector Operations)**.**

(1) *The* **zero vector**, *denoted $\vec{0}$, is $< 0, 0 >$.*
(2) *Let $\vec{a} =< a_1, a_2 >$ and $\vec{b} =< b_1, b_2 >$.*

$$\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2 >$$

(3) *Let c be a real number and $\vec{v} =< v_1, v_2 >$. We define* **scalar multiplication** *by*

$$c\vec{v} =< cv_1, cv_2 >$$

Definition 30. *Two vectors $\vec{x}$ and $\vec{y}$ are said to be* **parallel** *if $\vec{x} = c\vec{y}$ for some real number c. If two vectors are parallel and the constant c is negative, the two vectors are said to be* **antiparallel**.

Theorem 34 (Properties of the Vector Operations)**.** *If $\vec{x}$, $\vec{y}$, and $\vec{z}$ are vectors and a, b, c are scalars, then the following hold:*

(1) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ *(Associative Property)*
(2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ *(Commutative Property)*
(3) $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$ *(A Distributive Property)*
(4) $(a + b)\vec{x} = a\vec{x} + b\vec{x}$ *(Another Distributive Property)*
(5) $\vec{x} + \vec{0} = \vec{x}$
(6) $\vec{x} + (-\vec{x}) = \vec{0}$
(7) $1\vec{x} = \vec{x}$
(8) $0\vec{x} = \vec{0}$

Definition 31. *The* **standard basis vectors** *of the plane are $\vec{\imath} =< 1, 0 >$ and $\vec{\jmath} =< 0, 1 >$. A* **unit vector** *is any vector $\vec{u}$ such that $\| \vec{u} \| = 1$.*

Theorem 35 (Properties of the Norm)**.**

(1) $\| \vec{v} \| = 0$ *if and only if* $\vec{v} = \vec{0}$
(2) $\| c\vec{v} \| = |c| \, \| \vec{v} \|$
(3) $\| \vec{x} + \vec{y} \| \leq \| \vec{x} \| + \| \vec{y} \|$ *(The Triangle Inequality)*

Theorem 36. *Every nonzero vector is parallel to a unit vector.*

Definition 32. *A* **vector valued** *function $\vec{r}$ of one variable is a function that assigns to each t in a set A of real numbers a unique vector $\vec{r}(t)$ in the plane.*

Definition 33. *Let $\vec{r}(t)$ be a vector valued function. We write*

$$\lim_{t \to a} \vec{r}(t) = \vec{L}$$

if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |t-a| < \delta$ then $\| \vec{r}(t) - \vec{L} \| < \epsilon$.

Definition 34. *Let $\vec{s}$ be a vector valued function. Then the* **derivative** *of $\vec{s}$ at a point t is:*

$$\vec{s}\,'(t) = \lim_{h \to 0} \frac{1}{h} \left[\vec{s}(t+h) - \vec{s}(t) \right]$$

if the limit exists. $\vec{s}\,'$ is also called the **velocity** *at t. The speed is defined to be $\| \vec{s}\,'(t) \|$.*

Theorem 37. *Suppose $\vec{s}(t) =< x(t), y(t) >$ where x and y are real valued functions. If x and y are differentiable at t, then*

$$\vec{s}\,'(t) =< x'(t), y'(t) >$$

Theorem 38. *Suppose that the curve $y = f(x)$ is parameterized by the vector valued function $\vec{r}(t) =< x(t), y(t) >$, i.e. $y(t) = f(x(t))$. If x is a differentiable at t and f is differentiable at $x(t)$ and $x'(t) \neq 0$, then y is differentiable at $x(t)$ and*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$