Surgery and a Surgery

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Definitions I

- An n-manifold is a space locally the same as $\mathbf{R}^n$. Every point has a neighborhood homeomorphic to $\mathbf{R}^n$ or $\mathbf{R}^n_+$. There exists a basis of the space that has a countable number of elements.

- A knot is closed loop of rope. (embedding of the circle in $\mathbf{R}^3$)

- A solid torus is a donut (embedding of the donut into $\mathbf{R}^3$). A solid torus has a meridian and a longitude. (see slide to right).

Definitions II

- Note: Any curve on the boundary can be written as a sum of the meridian and the longitude. $C = p[m]+q[l]$ means "Go around in the meridian p times and go around the longitude q times"

- A solid torus neighborhood of a knot is a fattened up knot. It is a solid torus and has a meridian and a longitude.

- A link is a disjoint union of knots in $\mathbf{R}^3$.

Surgery

Basic idea the same as for doctors, cut out a heart, sew another heart back in and see what you get.

- Let M be a manifold, K a knot, U a solid torus neighborhood of K, and V a solid torus.

- Let $h : \partial V \to \partial U$. This is our "glue".

- The surgery on K in M is defined to be the quotient space $N = M - int(U) \cup_h V$

- Surgery is well-defined. We allways get a 3-manifold back and it doesn't depend on the size of the solid torus neighborhood. Also, it depends only a little on the "glue"

- The homeomorphism class of N depends only on the image class of the meridian, $[m] \to p[m] + q[l]$.

- We call the rational number $\frac{p}{q}$ the surgery coefficient of the knot.

- Thus, to specify a sugery on a 3-manifold, we need only specify the knot, the manifold and the surgery coefficient of the knot.

- The surgery coefficient correspnding to the map $[m] \to [m]$ is the 1/0 or ∞ surgery. On a knot, it doesn't change anything. The surgery coefficient corresponding to the map $[m] \to [l]$ is the 0/1 or the 0 surgery.

- We can also have surgeries on a link. For each knot in the link, we specify a surgery coefficient. Make sure that none of the solid torus neighborhoods intersect.

The 0-surgery on the Borromean Rings

Useful in classifying manifolds is the link known as the Borromean Rings.

On the right are three different representations of the Borromean rings.

The 0-surgery on the Borromean rings gives $S^1 \times S^1 \times S^1$. Before we perform this surgery, we need a lemma.

Lemma 1 *There exists an embedding* $\varphi : T(2) - int(U_1 \cup U_2) \to S^3 - int(H(2))$ *that is the identity on* $\partial T(2)$.

Proof: See Laudenbach and Roussaire.

Corollary 1 *If the surgery of index r is performed on the curves* C_1 *and* C_2 *in* $T(2)$, *the resulting space is homeomorphic to the* $1/r$ *surgery performed on* S^3-*int*($T(2)$). *The homeomorphism here is an extension of* φ

Proof: See Laudenbach and Roussaire

Theorem 1 *The 0 surgery on the Borromean rings is homeomorphic to* $S^1 \times S^1 \times S^1$.

Proof: The ouline of the proof is as follows:

- Split S^3 into two parts: a two holed torus that has the green curve on it's boundary and $S^3 - int$(two-holed torus) that contains the red curves.

- By our lemma, the 0 surgery on the two red curves in $S^3 - int$(two-holed torus) is homeomorphic to the 1/0 surgery on some curves in the two-holed torus.

- Deform the two-holed torus we got from the 0 surgeries and the two holed torus we took out.

- Identifying the two spaces on their boundaries (but not on the hole gives) $S^1 \times S^1 \times S^1$−(solid torus).

- This is homeomorphic to S^3−(solid torus neighborhood of green curve).

- This means that gluing in a solid torus along the boundary of $S^1 \times S^1 \times S^1$−(solid torus) by a homeomorphism which sends meridians to meridians is equivalent to gluing a solid torus to S^3−(solid torus neighborhood of green curve) which maps a meridian to a longitude).

- Thus the 0 surgery on the Borromean rings is homeomorphic to $S^1 \times S^1 \times S^1$.