# A COMPUTATIONAL TOOL FOR MAPPING CLASS GROUPS

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A major difficulty in the study of finite order mapping class groups lies in having useful ways to represent mapping classes. For surfaces, we know that every mapping class is a product of Dehn twists. Even the simplest periodic maps, however, have remarkably complicated representations as products of the Lickorish generators. A more palatable representation can be found amongst the matrix groups. More precisely, there exists a natural surjection from the full mapping class group of a surface of genus $g$ onto the integral symplectic group, $Sp(2g;\mathbb{Z}) \subset GL(2g;\mathbb{Z})$. The problem here is that there are elements of finite order in $Sp(2g,\mathbb{Z})$ that do not correspond to periodic diffeomorphisms on a surface of genus $g$.

These fake periodic maps can be exposed in the following way. Let $S$ be a smooth closed surface, $f : S \to S$ an orientation preserving diffeomorphism and $\Lambda(f)$ the Lefshetz number of $f$. Define $s : \mathbb{N} \to \mathbb{Z}$ to be the sequence $s(k) = \Lambda(f^k)$. If $f$ is periodic, the similarity class of $H(f) \in Sp(2g,\mathbb{Z})$ over $\mathbb{C}$ can be recovered from the sequence $s : \mathbb{N} \to \mathbb{Z}$. Thus, the task is to determine all $s : \mathbb{N} \to \mathbb{Z}$ that correspond to periodic maps on surfaces. This is done by establishing necessary and sufficient conditions on the Möbius inverse of $s : \mathbb{N} \to \mathbb{Z}$. The inverse sequence, denoted $M_s : \mathbb{N} \to \mathbb{Z}$, is defined by:

$$M_s(n) = \sum_{k|n} \mu(k) s(\frac{n}{k})$$

where $\mu$ is the Möbius function. In this talk, we will discuss the particulars of how to uncover fraudulent periodic maps and how it is rooted in A. Dold's general study of the sequence of Lefshetz numbers and its Möbius inverse.

The results are new, but invlove mostly elementary ideas with interesting visualizations. Their collection forms an extension of the work I presented in Fall 2005 to the algebraic topology seminar at the University of Hawai'i at Manoa. I would like to use an overhead projector during the talk.

# A Computational Tool for Mapping Class Groups

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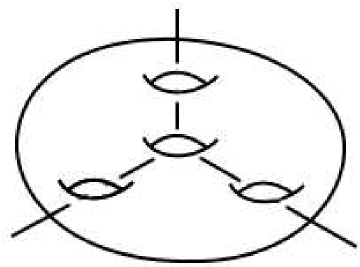
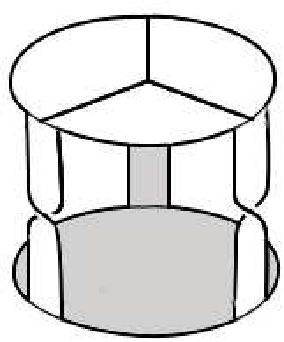
I. Examples of finite order maps on surfaces.

A map of order 3 on a surface of genus 4.



A map of order 3 on a surface of genus 1 (a Kosniowski generator).



<u>Conclusion:</u> Dehn twists are hard to use!

II. A short exact sequence.

$$(1) \to T(S) \xrightarrow{\alpha} Mod(S) \xrightarrow{\beta} Sp(2g, \mathbb{Z}) \to (1)$$

where $Sp(2g, \mathbb{Z})$ is the set of invertible matrices $A$ over the integers, such that $AJ\,{}^tA = J$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$\beta : \psi \to [H_1(\psi)]_\gamma,\ \psi \in Mod(S)$$

A Plus: Matrices are easy to deal with.
A Minus: There are elements in $Sp(2g, \mathbb{Z})$ of finite order that are not the image of an element of finite order in the mapping class group. For example:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Goal: To differentiate between the fakes and the real McCoy's by determining exactly the similarity classes over $\mathbb{C}$ which are the image of maps of finite order on a surface.

III. Lefshetz and Dold sequences: Using the algebraic topology.

<u>Defns:</u> Let $X$ be a Euclidean Neighborhood Retract (ENR), $f : X \to X$ a cont. map. The Lefshetz number is denoted $\Lambda(f)$. The Lefshetz sequence is the sequence:

$$(\Lambda(f), \Lambda(f^2), \Lambda(f^3), \ldots)$$

<u>Defn:</u> Let $s : \mathbb{N} \to \mathbb{Z}$ be any sequence. Define the Möbius inverse sequence of $s$ to be:

$$M_s(n) = \sum_{d|n} \mu(d) s(\frac{n}{d})$$

where $\mu$ is the Möbius function.

**Theorem 1 (Dold, 1983)** *If $Fix(f^k)$ is compact and $s : \mathbb{N} \to \mathbb{Z}$ is the Lefshetz sequence of $f$, then $k|M_s(k)$.*

Defn: A sequence $s : \mathbb{N} \to \mathbb{Z}$ is said to be a Dold sequence if $k|M_s(k)$ for all $k$. A Dold sequence is said to be periodic if there is an $m > 1$ such that $s(k+m) = s(k)$ for all $k$. The smallest such $m$ is called the period.

**Theorem 2 (MWC)** *If $s : \mathbb{N} \to \mathbb{Z}$ is a periodic Dold sequence of period $m$ and $k \not| m$, then $M_s(k) = 0$.*

**Corollary 3 (MWC)** *A periodic Dold sequence $s : \mathbb{N} \to \mathbb{Z}$ is completely determined by the numbers $s(k)$ where $k$ divides $m$.*

Note: This is the property that we will use to detect similarity classes. What is stated above is far more general than what is needed for periodic maps on surfaces.

## IV. Similarity classes and Dold sequences.

**Theorem 4** *If* $\psi : S \to S$ *is a periodic o.p. diffeomorphism of a surface* $S$ *of genus* $\geq 2$ *and* $s : \mathbb{N} \to \mathbb{Z}$ *is the Dold sequence of* $\psi$*, then the period of* $\psi$*. the period of* $s$*, and the period of* $\beta(\psi)$ *are all equal.*

Notation: Let $A \in Sp(2g, \mathbb{Z})$ have order $m$. If $\lambda$ is an eigenvalue of $A$, then all the roots of the minimal polynomial of $\lambda$ are also. $\lambda$ is a root of a $d$-th cyclotomic polynomial$\Phi_d(x)$, $d|m$. Let $m_d$ be the common multiplicity of the roots of $\Phi_d(x)$. Order them by size

$$d_1 = 1, d_2, \ldots, d_{\sigma_0(m)} = m$$

**Theorem 5 (MWC)** *Let $S$, $\psi$, $m$ and $s : \mathbb{N} \to \mathbb{Z}$ be as above and $G = \mathbb{Z}_m$. There is an integral matrix $B_G$ such that*

$$\begin{bmatrix} 2 - \Lambda(\psi) \\ 2 - \Lambda(\psi^{d_2}) \\ \vdots \\ 2 - \chi(S) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \mu(d_2) & \dots & \mu(m) \\ \dots & \dots & \dots & \dots \\ 1 & \varphi(d_2) & \dots & \varphi(m) \end{bmatrix}}_{B_G} \begin{bmatrix} m_1 \\ m_{d_2} \\ \vdots \\ m_{d_{\sigma_0(m)}} \end{bmatrix}$$

Examples: Let $p$, $q$ be primes.

$$B_{\mathbb{Z}_p} = \begin{bmatrix} 1 & -1 \\ 1 & p-1 \end{bmatrix}$$

$$B_{\mathbb{Z}_{pq}} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & (p-1) & -1 & 1-p \\ 1 & -1 & (q-1) & 1-q \\ 1 & (p-1) & (q-1) & (p-1)(q-1) \end{bmatrix}$$

Note: The left-hand side depends only on the important parts of the Dold sequence. So we just need to figure out what these are.

## V. Realizing the Dold sequences

**Theorem 6 (MWC)** *Let* $s : \mathbb{N} \to \mathbb{Z}$ *be a periodic Dold sequence of period* $m$*. Define:*

$$\Delta_s = 2m - (m+1)s(m) + \sum_{k|m} kM_s(k)$$

*Also, define* $K_s = \{k : M_s(k) \neq 0,\ 1 \leq k \leq m-1\}$*, where* $k \in K_s$ *is counted with multiplicity* $M_s(k)$*. Then the Dold sequence is realized if and only if it satisfies the following conditions:*

1. $s(m) \in -2\mathbb{N}$ *and if* $s(1) = 1$*, then* $K_s \backslash \{1\} \neq \emptyset$ *and* $\gcd(K_s \backslash \{1\}) = 1$*;*

2. $M_s(k) \geq 0$ *for all* $k \not\equiv 0 \pmod{m}$*;*

3. $\Delta_s \geq 0$ *and* $\Delta_s \equiv 0 \pmod{2m}$*;*

4. *If* $\Delta_s = 0$*, then* $\gcd(K_s) = 1$*.*