RESEARCH STATEMENT

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My current interests involve using algebraic and analytic number theory to investigate problems in algebraic and low-dimensional topology. These interests arise from competing aesthetic pleasures. On the one hand, I enjoy discovering theorems that distinguish interesting geometric properties of spaces. On the other hand, I can think of no better way to spend an afternoon than tinkering with Diophantine equations, congruence relations, or some elements of a cyclotomic field. I suppose that it is fortunate for me that these two competing interests work so well together. Currently, I am working on several projects that satisfy these passions.

Dold Sequences in Mapping Class Groups

The mapping class group of a closed orientable surface S is the group of orientation preserving diffeomorphisms of S modulo isotopies. By a famous theorem of Thurston [8], the elements of this group (called mapping classes) fall into three categories: periodic, reducible, or pseudo-Anosov. One difficulty of working with this group is that there are few efficient ways to represent the mapping classes. It is known that every mapping class can be written as a finite product of easily described maps called Dehn twists. However, it is hard to tell whether a given concatenation of these generators produces a periodic, reducible, or pseudo-Anosov map. Conversely, it is often very difficult to determine the Dehn twist decomposition of a given surface diffeomorphism.

Some relief is provided by the existence of a short exact sequence. Let S be a surface of genus g. Then we have:

$$(1) \to T(S) \xrightarrow{\alpha} \text{Mod}(S) \xrightarrow{\beta} \text{Sp}(2g,\mathbb{Z}) \to (1)$$

where $T(S)$ is the Torelli group of S, $\text{Mod}(S)$ is the mapping class group of S, and $\text{Sp}(2g,\mathbb{Z}) \subset \text{GL}(2g,\mathbb{Z})$ is the integral symplectic group. The map β is derived from the induced map of the mapping class in homology. The short exact sequence tells us that to some extent, we can attempt to study mapping classes via the matrix group $\text{Sp}(2g,\mathbb{Z})$ (and of course, information about the Torelli group). The catch is that this surjection can be misleading. For example, there are elements of finite order in $\text{Sp}(2g,\mathbb{Z})$ that cannot be the images of finite order mapping classes. This deviancy occurs even though the Torelli group contains no torsion for surfaces with genus at least 2!

The goal is to determine which elements of $\text{Sp}(2g,\mathbb{Z})$ necessarily correspond to images of each of the three possible types of mapping classes. For my dissertation, I used Dold sequences to find a partial answer to this question for finite order mapping classes. A Dold sequence is defined as follows. Let $s : \mathbb{N} \to \mathbb{Z}$ be an integer sequence. We associate to this sequence another sequence $M_s : \mathbb{N} \to \mathbb{Z}$ called the Möbius inversion sequence. This sequence is defined by the following equation:

$$M_s(n) = \sum_{d|n} \mu(d) s\left(\frac{n}{d}\right)$$

where μ is the Möbius function.

Let X be a topological space and $f : X \to X$ a continuous function. We will denote by $I(f)$ the fixed point index of f. We define a sequence $s(f) : \mathbb{N} \to \mathbb{Z}$ by $s(f)(n) = I(f^n)$. In 1983, A. Dold [3] proved that if X is a Euclidean Neighborhood Retract and the fixed point set of $f^n : X \to X$ is compact, then $n|M_{s(f)}(n)$. In the case of mapping classes of closed orientable surface, we have that $n|M_{s(f)}(n)$ for all $n \in \mathbb{N}$. It is thus natural to study the number theory of integer sequences $s : \mathbb{N} \to \mathbb{Z}$ such that $n|M_s(n)$ for all n. We will henceforth call such a sequence a Dold sequence.

My dissertation dealt primarily with Dold sequences which are also periodic. In this case, I showed that a Dold sequence $s : \mathbb{N} \to \mathbb{Z}$ is periodic with period m if and only if $M_s(n) = 0$ for all but finitely many n and $m = \text{lcm}\{n : M_s(n) \neq 0\}$. Moreover, this statement provides an elementary proof of the result of Babenko and Bogatyĭ[1] that a Dold sequence is bounded if and only if it is periodic. These theorems were then applied to the study of mapping class groups. Certainly, any periodic diffeomorphism has a periodic Dold sequence. The obvious question is: which periodic Dold sequences come from diffeomorphisms of periodic maps? The solution to this problem is contained in my dissertation[2]. Moreover, it is shown that this problem is equivalent to determining, up to similarity, which elements of $\text{Sp}(2g, \mathbb{Z})$ are images of finite order mapping classes. An algorithm for changing between interpretations of this problem is also provided.

Currently, I am investigating the pseudo-Anosov case using some theorems of Jiang and Guo [6]. I am also looking into the case of orientation preserving periodic maps on closed 3-manifolds. It is possible to use surgery theory to represent some of these maps as periodic maps on S^3 [7]. The idea is to use the surgery data to compute the sequence $s(f) : \mathbb{N} \to \mathbb{Z}$.

Applications of the Equivariant Signature to Surfaces

Let G be a cyclic group of order m acting on a closed orientable $2n$-manifold M as a group of orientation preserving diffeomorphisms. The g-signature, a generalization of the Hirzebruch signature, is an equivariant bordism invariant of such manifolds. It assigns to each $g \in G$ and each bordism class $[M]$ a complex number in $\mathbb{Z}[\lambda]$, where $\lambda = e^{2\pi i/m}$. The Atiyah-Singer g-signature theorem shows how to compute this algebraic number from the way in which the differential of g acts on the normal bundle of the fixed point set.

In the case of cyclic groups acting on surfaces, the Atiyah-Singer theorem reduces to a fairly usable expression. The fixed point set of $g \in G$ is simply a finite union of points. The normal bundle to each fixed point x is then identified with the tangent plane at that point and the differential acts as a rotation of that plane by some angle $2\pi j/m$, $\gcd(j, m) = 1$. The point x is associated to the algebraic number:

$$\alpha_{j/m} := \frac{\lambda^j + 1}{\lambda^j - 1}$$

The g-signature theorem states that if g has a_j fixed points corresponding to a rotation of $2\pi j/m$, then the signature is: $\sum_{k=1}^{m-1} a_j \alpha_{j/m}$. It is apparent that the algebraic number theory of the αs is of primary importance. In fact, it was proved in [4] that a maximally linearly independent subset of the αs is given by those j such that $1 \leq j \leq m/2$ and $\gcd(j, n) = 1$.

Robert Little, my dissertation adviser, and I have investigated the issue of independence in more detail when $m = p^n$, p a prime and $m > 2$. We established an explicit inductive relationship between the sets of αs from the p^{n-1} and p^n cases. It is then possible to use the independence in the $n = 1$ case to establish independence in the general case.

The motivation behind this investigation follows from Ewing [5]. In this paper on prime order group actions, it is proved that the equivariant signature map is "onto" if and only if the first factor of the ideal class group of $\mathbb{Q}(\lambda)$ is odd. The question is whether or not this relationship is coincidental or if it is due to some other underlying structure. Our investigation is an attempt to understand to what extent this discrepancy is caused by the algebraic number theory of the αs.

Closing Remarks

While I am preparing these results for publication, I am also developing some new projects. Recently, I have become interested in Lefschetz fibrations of symplectic 4-manifolds and Vassiliev invariants. I hope to be able to add something to these new and emerging fields.

Mathematical research is a part of my daily activities. It is important to remember, however, that one must maintain a balanced diet. I find that teaching and research complement each other. Each one has a positive impact on the other. Teaching mathematics, even at the most basic levels, serves to keep us honest and rigorous in our mathematical expositions. Conversely, the excitement of mathematical discovery can be passed to inquisitive students. It is therefore right and proper that these seemingly disparate activities be joined.

The topics listed above are my intended directions for the next several years. Yet, there is something to be said for the unexpected. I greatly enjoy working with other mathematicians and I hope that these collaborations will lead to new and surprising insights.

References

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