OVERVIEW OF DOLD SEQUENCES AND MAPPING CLASS GROUPS

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For a definition of Dold sequences, please view "Dold Sequences Overview" on my research site. Here we will briefly outline the applications of these sequences to mapping class groups of surfaces.

The mapping class group of a closed oriented surface S of genus g is the group of orientation preserving diffeomorphisms modulo isotopies. The greatest practical difficulty with these groups lies in the scarcity of ways to represent the mapping classes. There is a well-known set of generators for mapping class groups of surfaces called Dehn twists. While the theory of Dehn twists is very pretty and often very useful, it is quite difficult to characterize an arbitrary map in terms of them. For example, it is rather easy to come up with a periodic diffeomorphism of a surface. However, it is very difficult to write even the simplest ones as a composition of Dehn twists.

A nice "representation" of the mapping class group does exist. More precisely, let $\text{Mod}(S)$ denote the mapping class group of S and $\text{Sp}(2g, \mathbb{Z})$ the integral symplectic group. Define $\beta : \text{Mod}(S) \to \text{Sp}(2g, \mathbb{Z})$ to be the map which sends a mapping class to its induced map in integral homology and then writes it with respect to a fixed basis that behaves nicely with respect to the intersection form. It turns out that β is onto. Denote the kernel of β by $T(S)$ (also called the Torelli group). Thus we have the following short exact sequence:

$$(1) \to T(S) \xrightarrow{\alpha} \text{Mod}(S) \xrightarrow{\beta} Sp(2g, \mathbb{Z}) \to (1)$$

This is an improvement upon our original situation as it is easy to compute with matrices. However, information concerning $\text{Sp}(2g, \mathbb{Z})$ can be misleading about information in $\text{Mod}(S)$. For example, there are elements of finite order in $\text{Sp}(2g, \mathbb{Z})$ that are not the image of any element of finite order (and hence any diffeomorphism of finite order) in the mapping class group.

My idea is to study mapping class groups by getting a better handle on the deficiencies of the map β. In particular, I am trying to detect these deficiencies using the following integer sequence:

$$(\Lambda(\psi), \Lambda(\psi^2), \Lambda(\psi^3), \ldots)$$

where $\psi \in \text{Mod}(S)$ and $\Lambda(\psi^k)$ is the rational Lefshetz number of ψ^k. Now, the Lefshetz number of $\psi \in \text{Mod}(S)$ is easy to compute. It's just $\Lambda(\psi) = 2 - Tr(H_1(\psi))$. Thus the Lefshetz sequence determines the traces of powers of $H_1(\psi)$. This in turn reveals information about the characteristic polynomial of the induced map. All of this should be encoded in the number theory of Dold sequences. So we ask: Which Dold sequences are the Lefshetz sequences of mapping classes in $\text{Mod}(S)$?

Now, Thurston classified the mapping classes of hyperbolic surfaces. His theorem says that a mapping class is either of finite order, reducible, or pseudo-Anosov. Knowing that a Dold sequence on a Euclidean Neighborhood retract is either periodic or asymptotic to e^x, it is reasonable to attack our classification of Dold sequences along the same lines as the Thurston classification. Thusfar, I have determined all of the periodic Dold sequences that are the Lefshetz sequences of finite order mapping classes. I have also discovered an algorithm that returns the eigenvalues of the induced map from its Lefshetz sequence. Recently, I have been developing some ideas about the pseudo-Anosov case.

If you would like more information about my applications of Dold sequences to mapping class groups, please e-mail me at the above address.