OVERVIEW OF FINITE CYCLIC ACTIONS ON HANDLEBODIES

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A handlebody is a compact, connected, orientable three dimensional manifold whose boundary is a closed, connected, and orientable surface (called a "surface" in the sequel). Suppose that M is a handlebody and G is a group acting on M as a group of orientation preserving diffeomorphisms that maps the boundary smoothly to itself. Clearly, the restriction of the action to ∂M gives a group action on the bounding surface. Now, every orientable surface bounds a handlebody. If we have a smooth action G on a surface S, under what conditions does that action extend to an action on the handlebody?

We could ask the same question using the phraseology of equivariant bordism: Which smooth actions on a surface are equivariant boundaries? If one wanted to show that a given action was not an equivariant boundary, one could use an invariant called the equivariant signature. This invariant, due to Atiyah and Singer, can be easily shown to vanish on equivariant boundaries. The goal is to use this invariant to say things in the other direction: Are there any smooth actions with vanishing equivariant signature that do not extend to the handlebody?

As part of my thesis, I considered the simplest case. Let $G = \mathbb{Z}_p$, p an odd prime, and supppose that S is a surface admitting a smooth orientation preserving G action. Each fixed point of the action is a fixed point of any generator of G. Fix a generator $\psi : S \to S$ of G. Now, the group G can be considered as a group of automorphisms of a Riemann surface. Hence we can choose a coordinate neighborhood of each fixed point such that ψ is just the map $z \to e^{2\pi i j/p} z$, where $i = \sqrt{-1}$ and $1 \leq j \leq p-1$. Let $\lambda = e^{2\pi i/p}$ and define:

$$\alpha_{j/p} = \frac{\lambda^j + 1}{\lambda^j - 1}$$

For each fixed point of $\psi : z \to \lambda^j z$, we associate the algebraic number $\alpha_{j/p}$. According to Atiyah and Singer, the G-signature of ψ can be expressed as a sum of the $\alpha_{j/p}$'s:

$$\sigma(\psi) = \sum_{\text{fixed pts.}} \alpha_{j/p}$$

Hence, to study the G-signature of smooth actions on surfaces, it is necessary to learn something about linear combinations of the $\alpha_{j/p}$'s. Bob Little (my thesis adviser) has shown that the following set is linearly independent over $\mathbb{Q}$:

$$\{\alpha_{1/p},\, \alpha_{2/p}, \ldots,\, \alpha_{\mu/p}\},$$

where $\mu = (p-1)/2$.

I discovered some consequences of this result that were quite surprising. For example, we have:

Theorem 1. *A smooth $\mathbb{Z}_p$ action on a surface S extends to the handlebody if and only if $\sigma(\psi) = 0$, where ψ is any generator of the action.*

Ideally we would like to do the same kind of thing with arbitrary finite cyclic actions. The algebraic number theory in this case quickly humbled us. This independence problem was posed to several specialists and they echoed our concerns. Much more work on both the algebraic number theory and the equivariant topology sides of this problem needs to be done.

If you would like more information about this topic, please e-mail me at the address above.