Definition of G-Signature

1. A is a finitely generated free $\mathbb{Z}$-module of dimension n.

2. $G = \mathbb{Z}_m$, $m \geq 2$, acts on A as a group of isomorphisms.

3. $\Phi : A \otimes A \to \mathbb{Z}$ a symmetric or skew-symmetric, unimodular, G-invariant, bilinear form.

4. $V = \mathbb{C} \otimes A$. $< \cdot, \cdot >$ a Hemitian G-invariant inner product on V.

$$g(\alpha \otimes x) = \alpha \otimes g \cdot x$$

$$\Phi_{\mathbb{C}}(\alpha \otimes x, \beta \otimes y) = \alpha \Phi(x, y) \bar{\beta}$$

5. Let g be a generator of G. $\theta_g : V \to V$, $g \to g \cdot x$. $\theta_g^m = 1$. Eigenvalues of θ_g are powers of $\lambda = e^{2\pi i/m}$. Let V_j be the eigenspace of λ^j.

6. $V = \oplus_{k=0}^{m-1} V_j$, since minimal polynomial of θ_g has roots of multiplcity 1.

7. Define $L : V \to V$ by $< Lu, v >= \Phi_{\mathbb{C}}(u \otimes v)$.

8. If Φ is symmetric, $LV_j \subset V_j$, L is self-adjoint (and hence normal), and L has only real eigenvalues.

9. If Φ is skew-symmetric, $LV_j \subset V_j$, $-iL$ is skew-adjoint (and hence normal) and $-iL$ has only real eigenvalues.

10. By the Spectral Theorem, V_j is the direct sum of eigenspaces of L (or $-iL$). $ker(L) = 0$, since Φ is unimodular. V_j^+ is the subsbace corresponding to positive eigenvalues of L (or $-iL$, V_j^- is subspace corresponding to negative eigenvalues of L(or $-iL$).

11. The g-signature is defined to be:

$$\sigma(g, A, \Phi) = \sum_{j=0}^{m-1} (dim_{\mathbb{C}}(V_j^+) - dim_{\mathbb{C}}(V_j^-))\lambda^j$$

<u>Example</u>: Let $M = S^1 \times S^1$, $A = H_1(M;\mathbb{Z})$. Let Φ be the intersection number. The mapping class group of the torus, modulo isotopies is $GL(2,\mathbb{Z})$. Let g be represented by the matrix:

$$\theta_g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Lemma 1 *With the conventions established above, we have the following:*

1. *The minimal polynomial of θ_g divides $x^m - 1$ but does not necessarily equal $x^m - 1$.*

2. *θ_g is diagonalizable and for each eigenvalue of θ_g, the geometric multiplicity equals the algebraic multiplicity.*

3. *Let $m_j \geq 0$ be the multiplicity of λ^j as an eigenvalue of θ_g for $0 \leq j \leq m-1$. If λ^k is a conjugate of λ^j, then $m_j = m_k$.*

4. *Let* $\Phi_d(x)$ *be the* d*-th cyclotomic polynomial and let* $\phi_j(x) = min_{\lambda^j}(x)$*. Then if* $m_j \neq 0$, $\phi_j(x) = \Phi_d(x)$ *for some* $d|m$*. Define* $m_d \geq 0$ *to be the common multiplicity of all the roots of* $\Phi_d(x)$ *in the characteristic polynomial. Thus, the characteristic polynomial of* θ_g *is:*

$$char_{\theta_g}(x) = \prod_{d|m} (\Phi_d(x))^{m_d}$$

5. *There is a relationship between* n, m *and the multiplicities* m_d*:*

$$n = \sum_{d|m} m_d \varphi(d)$$

where φ *is the Euler* φ *function.*

Theorem 2 (Berend-Katz) *Let A be a finitely generated $\mathbb{Z}$-module with a finite group G acting as a group of isomorphisms. Let Φ be a unimodular symmetric bilinear form on A. Then:*

1. $Sign(g, A, \Phi) \in \mathbb{Z} + 2\mathbb{Z}[\lambda]$
2. *If Φ is either positive definite or negative definite, then $Sign(g, A, \Phi)$ is an integer.*

$\underline{Definition:}$ A bilinear form β on an R-module X is said to be an *inner product* if for all R-linear maps $\varphi : X \to R$, there exists exactly one x_0 and exactly one y_0 such that $\beta(x_0, \_) = \varphi$ and $\beta(\_, y_0) = \varphi$. Equivalently, each of the following maps from $X \to Hom(X, R)$ is a bijection:

$$x \to \beta(x, \_)$$

$$y \to \beta(\_, y)$$

$\underline{Definition:}$ A bilinear form module homomorphism from (X, β) to (X', β') is a R-module homomorphism $f : X \to X'$ such that for all $x, y \in X$:

$$\beta(x, y) = \beta'(f(x), f(y))$$

Proposition 1 *If (X, β) is an inner product module, (X', β') a bilinear form module and $(X, \beta) \cong (X', \beta')$, then (X', β') is an inner product module.*

$\underline{Definition}:$ If (X, β) is an inner product module and X is finitely generated of R and projective, then we will call the pair an $inner\ product\ space$.

Let X be a finitely generated R-module with basis $e_1, \ldots, e_n$ and β a bilinear form on X. Let B be the $n \times n$ matrix with entries $\beta(e_i, e_j)$

Lemma 2 (X, β) *is an inner product space if and only if* B *is invertible.*

G a group. A a f.g. free $\mathbb{Z}$-module. $V = \mathbb{C} \otimes A$.

$$\rho : G \to GL(V)$$

a representation of G. Consider triples (V, β, ρ)

- β is a Hermitian or skew-Hermitian inner-product on V.

- β is ρ-invariant .

Goal: Give triples (V, β, ρ) the structure of a commutative ring with identity.

1. Addition: (V_1, β_1, ρ_1), (V_2, β_2, ρ_2). Their sum is $(V_1 \oplus V_2, \beta_1 \oplus \beta_2, \rho_1 \oplus \rho_2)$ where:

$$\beta_1 \oplus \beta_2 : (V_1 \oplus V_2) \times (V_1 \oplus V_2) \to \mathbb{C}$$

$$\beta_1 \oplus \beta_2 (x_1 \oplus x_2, y_1 \oplus y_2) = \beta_1(x_1, y_1) + \beta_2(x_2, y_2)$$

Note: The cup product induces a bilinear form AFTER you compose with the inclusion of the cross product into the tensor prodcut. The literature tends to leave out this detail.

$$\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2)$$

$$\rho_1 \oplus \rho_2(g) = \rho_1(g) \oplus \rho_2(g)$$

$$g(x \oplus y) = gx \oplus gy$$

Lemma 1 *Given triples* (V_1, β_1, ρ_1), (V_2, β_2, ρ_2),

(a) $\rho_1 \oplus \rho_2 \in GL(V_1 \oplus V_2)$

(b) $\beta_1 \oplus \beta_2$ *is an inner product*

(c) $\beta_1 \oplus \beta_2$ *is* $\rho_1 \oplus \rho_2$*-invariant.*

(d) $V_1 \oplus V_2 \cong V_2 \oplus V_1$

2. Multiplication:
   Given triples (V_1, β_1, ρ_1), (V_2, β_2, ρ_2), their product is $(V_1 \otimes V_2, \beta_1 \otimes \beta_2, \rho_1 \otimes \rho_2)$ where

$$\beta_1 \otimes \beta_2 : (V_1 \otimes V_2) \times (V_1 \otimes V_2) \to \mathbb{C}$$

$$\beta_1 \otimes \beta_2(x_1 \otimes x_2, y_1 \otimes y_2) = \beta_1(x_1, y_1)\beta_2(x_2, y_2)$$

$$\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$$

$$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g)$$

$$g(x \otimes y) = gx \otimes gy$$

Lemma 2 *Given triples* (V_1, β_1, ρ_1), (V_2, β_2, ρ_2), *with* ρ_1 ϵ_1*-Hermitian and* ρ_2 ϵ_2*-Hermitian,*

(a) $\rho_1 \otimes \rho_2 \in GL(V_1 \otimes V_2)$

(b) $\beta_1 \otimes \beta_2$ *is an inner product*

(c) $\beta_1 \otimes \beta_2$ *is* $\rho_1 \otimes \rho_2$*-invariant.*

(d) $\rho_1 \otimes \rho_2$ *is* $\epsilon_1\epsilon_2$*-Hermitian.*

(e) $V_1 \otimes V_2 \cong V_2 \otimes V_1$

3. Unit: Let $< u >= (\mathbb{C}e_1, \alpha, \tau)$ where $u = \pm 1$, $\beta(e_1, e_1) = u$, and $\tau : G \to \mathbb{C}^*$ is the trivial representation:

$$\tau(g) = 1 \, \forall g \in G$$

Lemma 3 *For all triples* (V, β, ρ)*,*

$$< 1 > \otimes (V, \beta, \rho) \cong (V, \beta, \rho) \otimes < 1 > \cong (V, \beta, \rho)$$

4. $\underline{Witt Equivalence}$

Definition: Let (V, β, ρ) be a triple and $W \subset V$ a ρ invariant submodule. Then the $orthogonal$ $complement$ $W^{\perp}$ is the ρ-invariant submodule:

$$\{v \in V : \beta(w, v) = 0 \,\text{for all} w \in W\}$$

Note that (W, β, ρ) may not itself be a triple.

Definition: A triple (V, β, ρ) is said to be $split$ if V contains a ρ-invariant submodule W such that $W^{\perp} = W$.

Lemma 4 *Let* (S_1, β_1, ρ_1) *and* (S_2, β_2, ρ_2) *be split and let* (V, β, ρ) *be any triple. Then:*

(a) $< 1 > \oplus < -1 >$ *is split.*

(b) $S_1 \oplus S_2$ *is split.*

(c) $S_1 \otimes V$ *is split.*

(d) $(V, \beta, \rho) \oplus (V, -\beta, \rho)$ *is split.*

Definition: Two triples (V_1, β_1, ρ_1) and (V_2, β_2, ρ_2) are said to be $Witt\ equivalent$ if there exists split triples (S_1, α_1, τ_1) and (S_2, α_2, τ_2) such that:

$$V_1 \oplus S_1 \cong V_2 \oplus S_2$$

Theorem 5 *The set of equivalence classes of triples under Witt equivalence form a commutative ring with identity.*

- M^{2n} a closed, orientable, $2n$-dim. manifold.

- $g : M \to M$ a orientation preserving diffeomorphism of finite order m.

- Let M^g be the fixed point set of g. ν the normal bundle of the f.p. set of M^g in M. Assume: ν has a complex structure that is compatible with the g-action.

- Over each connected component of M^g, ν is the direct sum of subbundles ν_j where g acts as multiplication by λ^j, $0 < j < m$.

- μ_j is the complex dimension of ν_j. For $x \in M^g$, associate an $(m-1)$-tuple (called the normal slice type of x):

$$\mu = (\mu_1, \ldots, \mu_{m-1})$$

- M^g_μ is the set of points in M^g with normal slice type μ

<u>Bilinear Form:</u> Let $A = H^n(M;\mathbb{Z})/Tor$.

$$\Phi : A \times A \to H^{2n}(M;\mathbb{Z}) \cong \mathbb{Z}$$

$$(a,b) \to < a \cup b, [\xi_M] >$$

Theorem 1 (Atiyah-Singer)

$$\sigma(g, A, \Phi) = \sum_\mu \prod_j \prod_{l=1}^{\mu_j} \frac{\lambda^j e^{2x_{j,l}} + 1}{\lambda^j e^{2x_{j,l}} - 1} \mathscr{L}(M^g_\mu)[M^g_\mu]$$

where $\{x_{j,l}\}$ *are the* $2-d$ *Wu classes of* ν_j*,* $\mathscr{L}(M^g_\mu)$ *is the Hirzebruch* $\mathscr{L}$*-class, and* $[M^g_\mu]$ *denotes the preferred fundamental cycle of* M^g_μ*.*

Let p be an odd prime, $M = S_j$. Let P denote the set of fixed points of the $\mathbb{Z}_p$-action.

The slice type of each $x \in P$ is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$. The above reduces to:

$$\sigma(g, A, \Phi) = \sum_{\mu} \frac{\lambda^j + 1}{\lambda^j - 1}$$

$$\sigma(g, H^1(S_j), \Phi) = q\frac{\lambda + 1}{\lambda - 1} + \frac{\lambda^j + 1}{\lambda^j - 1}$$

Let $\alpha_j = (\lambda^j + 1)/(\lambda^j - 1)$ and let $\{\alpha_1, \ldots, \alpha_{m-1}\}$ be the $\mathbb{Z}$-module generated by the α_j's.

Lemma 2 *If M^{2n} is an equivariant boundary, then $(H^n(M; \mathbb{Z}), \Phi, \rho)$ is split.*

Let p be an odd prime. Define:

$$ab_{2n} : \mathscr{O}_{2n}(\mathbb{Z}_p; \mathbb{Z}) \to W_{2n}(\mathbb{Z}_p; \mathbb{Z})$$

Then $\sigma \circ ab_{2n} : \mathscr{O}_{2n} \to \mathbb{Z}[\lambda]$. Today: $n = 1$. Let $m = (p-1)/2$. Define:

$$\mathscr{M}_2 = im(\sigma \circ ab_2)$$

$$\mathscr{R}_2 = \left\{ \sum_{k=1}^{m} a_i(\lambda^k - \lambda^{-k}) | a_1 \equiv \ldots \equiv a_m (mod 2) \right\}$$

Lemma 3 $\mathscr{M}_2 \subset \mathscr{R}_2$

Lemma 4 $\mathscr{M}_2 \subset \{\alpha_1, \ldots, \alpha_{m-1}\}$

Lemma 5 $\mathscr{M}_2 = \{\alpha_1, \ldots, \alpha_{m-1}\} \cap \mathscr{R}_2$

<u>Remark:</u> From the above lemma, we can conclude that $x \in \{\alpha_1, \ldots, \alpha_{m-1}\}$ is in $\mathscr{M}_2$ if and only if x is an algebraic integer.

Theorem 1 *Let $\mathbb{A}$ be the set of algebraic integers in $\mathbb{C}$, p and odd prime, and $\lambda = e^{2\pi i/p}$. Then $\mathbb{A} \cap \mathbb{Q}[\lambda] = \mathbb{Z}[\lambda]$.*

Lemma 2 *Let $\{\gamma_1, \ldots, \gamma_n\}$ be a basis for a number field K over $\mathbb{Q}$ consisting entirely of algebraic integers and let $d = disc(\gamma_1, \ldots, \gamma_n)$. Then every $\gamma \in \mathbb{A} \cap K$ can be expressed as:*

$$\gamma = \frac{m_1\gamma_1 + \ldots + m_n\gamma_n}{d}$$

where $m_j \in \mathbb{Z}$ and $d | m_j^2$ for all j.

Lemma 3 $\mathbb{Z}[1-\lambda] = \mathbb{Z}[\lambda]$ *and* $disc(1-\lambda) = disc(\lambda)$

Lemma 4 $disc(\lambda) = p^{p-2}$

Lemma 5 $p = \prod_{k=1}^{p-1}(\lambda^k - 1)$

Lemma 6 *Let $\alpha_j = (\lambda^j + 1)/(\lambda^j - 1)$. Then:*

$$\alpha_j = \frac{1}{p} \sum_{k=1}^{m} (2k - p)(\lambda^{jk} - \lambda^{-jk})$$

Corollary 7 *(Ewing) $x \in \{\alpha_1, \ldots, \alpha_m\}$ is in $\mathscr{M}_2$ if and only if x is an algebraic integer.*

Corollary 8 *For $2 - d$ manifolds, $\sigma \circ ab_*$ (when defined) is purely imaginary or 0.*

<u>Note:</u> $\alpha_j^{-1} = 2\alpha_{2j} - \alpha_j$ is an algebraic integer but α_j is not.