Number Theory Problem Set #2

Numerical Problems
1. Make addition tables for \( \mathbb{Z}_5 \) and \( \mathbb{Z}_6 \). Make multiplication tables for \( \mathbb{Z}_5 \), \( \mathbb{Z}_6 \), \( \mathbb{Z}_9 \), and \( \mathbb{Z}_{11} \). Do you notice any patterns?

2. ⊛ Which of the following elements can you find in \( \mathbb{Z}_5 \)?
   
   \[ 4 \cdot 5, \ 2 - 6, \ 1/2, \ 2/5, \ \sqrt{2}, \ \sqrt{-3}, \ \sqrt{-1}, \ \sqrt{6} \]

3. Which of the following can you find in \( \mathbb{Z}_{11} \)? In \( \mathbb{Z}_{35} \)? In \( \mathbb{Z}_9 \)? Any conjectures?
   
   \[ 7 + 8, \ 4 - 9, \ 3 \cdot 5, \ 5^2, \ 5^3, \ 1/5, \ 3/8, \ \sqrt{3}, \ \sqrt{-2}, \ \sqrt{-6} \]

4. Find a solution to each of the following linear Diophantine equations (your work on Problem Set #1 might be helpful):
   
   \[ 16x + 5y = 1, \ \ 29x + 11y = 1, \ \ 49x + 36y = 1. \]

Exploration
5. Consider the complex numbers \( a + bi \) with \( a \) and \( b \) in \( \mathbb{Z} \) (\( i = \sqrt{-1} \)). These numbers form under addition and multiplication a mathematical system not unlike \( \mathbb{Z} \). This system is usually denoted \( \mathbb{Z}[i] \) and is called the Gaussian integers. Practice adding and multiplying some examples of these numbers to get used to how it works. Which of the axioms we have developed so far hold for \( \mathbb{Z}[i] \)?

Prove or Disprove and Salvage if Possible (PODASIP)
6. ⊛ \( a \cdot 0 = 0 \) for every \( a \in \mathbb{Z} \).

7. \( -1 \cdot a = -a \) for every \( a \in \mathbb{Z} \).

8. \( -(−a) = a \) for every \( a \in \mathbb{Z} \).

9. If \( d|a \) and \( d|b \), then \( d|(ax + by) \) for all integers \( d \), \( x \) and \( y \).

10. ⊛ If \( a|b \), then \( a \leq b \) for all \( a, b \in \mathbb{Z} \).

11. ⊛ \( 1 \in \mathbb{N} \).