Brooke L-fcns (joint w/ K. Martin & D. Whitehouse)

Waldspurger: E/F quad ext
η quad character
π cuspidal automorphic rep'n of $PGL_2(F)$

$L(s,\pi_E) = L(s,\pi)L(s,\pi\otimes\eta)$
π_E = base change (π)

Functional equation:

$X(E,\pi) = \left\{ \begin{array}{l}\text{quaternion}\\ \text{algebras}\end{array} D/F : E \hookrightarrow D \text{ and } \begin{array}{l}\exists \pi^D \in D^\times \text{ s.t.}\\ J.L.(\pi^D) = \pi\end{array} \right\}$

Thm (W): $L(\tfrac{1}{2},\pi_E) = 0 \iff \int_{\mathbb{A}_F^\times E^\times \backslash \mathbb{A}_E^\times} \phi(t)\,dt = 0 \quad \forall \phi \in \pi^D \quad \forall D \in X(E,\pi)$

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^s$$
$$= \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$

$$E(i,s) = \sum_{(c,d)=1} \frac{1}{(c^2+d^2)^s} \qquad \zeta(2s) = \sum_{(c,d)\neq(0,0)} \frac{1}{(c^2+d^2)^s} = \sum_{(c,d)\neq(0,0)} \frac{1}{N_{\mathbb{Q}(i)/\mathbb{Q}}(ci+d)^s}$$
$$= \zeta_{\mathbb{Q}(i)}(s)$$

$$|\text{Periods of } E(z,s)|^2 = \Big|\sum_{\substack{z\in CM_d\\ d=\text{neg fund. disc.}}} E(z,s)\,\zeta(2s)\Big|^2 = \zeta_{\mathbb{Q}(\sqrt{d})}(s)^2$$

E/F split at arch. places
(technical - needed for current proof but maybe not nec.)

π cuspidal auto. rep'n of $PGL(2n,\mathbb{A}_F)$
- s.c. at a finite place that splits in E

π^D rep'n of $PGL(n, D(\mathbb{A}_F))$

Thm: If $\exists \varphi \in \pi^D$ s.t. $\int_{PGL(n,E)\backslash PGL(n,\mathbb{A}_E)} \varphi(t) \neq 0$ then $L(1/2,\pi^E) \neq 0$

"Idea of Pf"
Waldspurger Θ-corresp original proof, reproved by Jacquet by comparing two relative trace formulas

$$H = PGL(n,F)\times PGL(n,F) \hookrightarrow G = PGL(2n,F)$$

$$H' = PGL(n,E) \hookrightarrow G' = PGL(n,D)$$

$$L(1/2, \pi_E) = L(1/2, \pi)\, L(1/2, \pi\eta)$$

$\int\int$ Friedberg-Jacquet

$$\sum_{\pi\in\mathcal{A}(G)} \sum_{\varphi\in \text{o.n.b.}(\pi)} \int_H \varphi \int_H \varphi(h)\,\eta(\det h)$$

$$\overset{\text{Rel. Tr. Form}}{=} \sum_{H\backslash G/H} \int_{H\times H} f(h_1\gamma h_2)\,\eta(\det h)$$

$$\sum_{\pi'} \sum_{\varphi'} \left| \int_{H'} \varphi'(h) \right|^2 \overset{\text{R.T.F.}}{=} \sum_{H'\backslash G'/H'} \int_{H'\times H'} f'(\ \ldots)$$