Computations with Coleman integrals

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AWM 40 Years and Counting (Number Theory Session)
Saturday, September 17, 2011
Outline

1 Introduction
   - $p$-adic (Coleman) integrals
   - Hyperelliptic curves

2 Results
   - Single integrals (joint with Bradshaw, Kedlaya)
   - Integrals that compute heights (joint with Besser)
   - Application: a $p$-adic BSD conjecture (joint with Müller, Stein)

3 Conclusion
   - Future work
What can Coleman integrals tell us?

Different integrals, different things:

- Experiments with Chabauty’s method: find \( P \) such that \( \int_0^P \omega = 0 \)
- Torsion points on curves (Coleman’s original application, for curves of \( g > 1 \))
- Kim’s nonabelian Chabauty method: use \( \int_b^z \omega_0 \omega_1 \) to recover integral points on elliptic curves
- \( p \)-adic heights on curves: \( h_p(D_1, D_2) = \int_{D_2} \omega_{D_1} \)
- Syntomic regulators on curves: for \( \{f, g\} \in K_2(C) \),
  \( \text{reg}_p(\{f, g\})(\omega) = \int_{\{f\}} \log(g) \omega \)
- \( p \)-adic polylogarithms and multiple zeta values, following Besser-de Jeu
Why hyperelliptic curves?

A hyperelliptic curve is a smooth projective curve $C/K$ (char $K \neq 2$) of genus $g \geq 1$ with an affine model

$$y^2 = f(x),$$

with $\deg f(x) \leq 2g + 2$.

Why hyperelliptic curves?

- Key step in the calculation of Coleman integrals: matrix of Frobenius on de Rham cohomology
- Kedlaya’s algorithm does this (and has been implemented) for hyperelliptic curves, so hyperelliptic curves are a natural starting point

Analogues of Kedlaya’s algorithm exist for other classes of curves and varieties, so what we’re describing today could be extended to these objects (more later)
Results

Notation and setup

- $X$: genus $g$ hyperelliptic curve (of the form $y^2 = f(x)$, $f$ monic of degree $d$) over $K = \mathbb{Q}_p$
- $p$: prime of good reduction
- $\overline{X}$: special fibre of $X$
- $X^\text{an}_{C_p}$: generic fibre of $X$ (as a rigid analytic space)
Results

Notation and setup, in pictures

- There is a natural reduction map from $X^\text{an}_{\mathbb{C}_p}$ to $\overline{X}$; the inverse image of any point of $\overline{X}$ is a subspace of $X^\text{an}_{\mathbb{C}_p}$ isomorphic to an open unit disk. We call such a disk a residue disk of $X$.

- A wide open subspace of $X^\text{an}_{\mathbb{C}_p}$ is the complement in $X^\text{an}_{\mathbb{C}_p}$ of the union of a finite collection of disjoint closed disks of radius $\lambda_i < 1$:

\begin{align*}
X^\text{an}_{\mathbb{C}_p} & \quad \text{red} \quad \overline{X} \\
\text{red}^{-1}(P) & \quad \text{red}^{-1}(R) \\
\text{red}^{-1}(S) & \quad \text{red}
\end{align*}
Computing tiny integrals

We refer to any Coleman integral of the form $\int_P^Q \omega$ in which $P, Q$ lie in the same residue disk as a *tiny integral*. To compute such an integral:

- Construct a linear interpolation from $P$ to $Q$. For instance, in a non-Weierstrass residue disk, we may take

$$
x(t) = (1 - t)x(P) + tx(Q)
$$

$$
y(t) = \sqrt{f(x(t))},
$$

where $y(t)$ is expanded as a formal power series in $t$.

- Formally integrate the power series in $t$:

$$
\int_P^Q \omega = \int_0^1 \omega(x(t), y(t)) \, dt.
$$
Properties of the Coleman integral

Coleman formulated an integration theory on wide open subspaces of curves over $\mathcal{O}$, exhibiting no phenomena of path dependence. This allows us to define $\int^Q_P \omega$ whenever $\omega$ is a meromorphic 1-form on $X$, and $P, Q \in X(Q_p)$ are points where $\omega$ is holomorphic. Properties of the Coleman integral include:

**Theorem (Coleman)**

- **Linearity:** $\int^Q_P (\alpha \omega_1 + \beta \omega_2) = \alpha \int^Q_P \omega_1 + \beta \int^Q_P \omega_2$.
- **Additivity:** $\int^R_P \omega = \int^Q_P \omega + \int^R_Q \omega$.
- **Change of variables:** if $X'$ is another such curve, and $f : U \to U'$ is a rigid analytic map between wide opens, then $\int^Q_P f^* \omega = \int_{f(P)}^{f(Q)} \omega$.
- **Fundamental theorem of calculus:** $\int^Q_P df = f(Q) - f(P)$.
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Coleman’s construction

How do we integrate if $P, Q$ aren’t in the same residue disk? Coleman’s key idea: use Frobenius to move between different residue disks (Dwork’s “analytic continuation along Frobenius”)

So we need to calculate the action of Frobenius on differentials.
Frobenius, MW-cohomology

- $X'$: affine curve ($X - \{\text{Weierstrass points of } X\}$)
- $A$: coordinate ring of $X'$

To discuss the differentials we will be integrating, we recall: The *Monsky-Washnitzer (MW) weak completion of $A$* is the ring $A^\dagger$ consisting of infinite sums of the form

$$\left\{ \sum_{i=-\infty}^{\infty} \frac{B_i(x)}{y^i}, \ B_i(x) \in K[x], \deg B_i \leq 2g \right\},$$

further subject to the condition that $v_p(B_i(x))$ grows faster than a linear function of $i$ as $i \to \pm \infty$. We make a ring out of these using the relation $y^2 = f(x)$.

These functions are holomorphic on wide opens, so we will integrate 1-forms

$$\omega = g(x,y) \frac{dx}{2y}, \ g(x,y) \in A^\dagger.$$
Odd vs even degree models of the curve

Any odd differential $\omega = g(x, y) \frac{dx}{2y}, g(x, y) \in A^\dagger$ can be written as

$$\omega = df_\omega + c_0 \omega_0 + \cdots + c_{d-2} \omega_{d-2},$$

where $f_\omega \in A^\dagger, c_i \in K$ and

$$\omega_i = \frac{x^i \, dx}{2y} \quad (i = 0, \ldots, d - 2),$$

with $d = \deg f(x)$. When $d = 2g + 1$, the set $\{\omega_i\}_{i=0}^{d-2}$ forms a basis of the odd part of the de Rham cohomology of $A^\dagger$. Let us suppose that $d$ is odd.

By linearity and the fundamental theorem of calculus, we reduce the integration of $\omega$ to the integration of the $\omega_i$. 
Integrals between points in non-Weierstrass disks

Let $\phi$ denote Frobenius. Recall that a *Teichmüller point* of $X$ is a point $P$ such that $\phi(P) = P$.

One way to compute Coleman integrals $\int_P^Q \omega_i$:

- Find the Teichmüller points $P', Q'$ in the residue disks of $P, Q$.
- Use Frobenius to compute $\int_{P'}^Q \omega_i$.
- Use additivity in endpoints to recover the integral:  
  $$\int_P^Q \omega_i = \int_P^{P'} \omega_i + \int_{P'}^Q \omega_i + \int_{Q'}^Q \omega_i.$$
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More on Frobenius:

- Calculate the action of Frobenius $\phi$ on each basis differential, letting

$$\phi^* \omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij} \omega_j.$$ 

- Compute $\int_{P'} Q \omega_j$ by solving a linear system.

- As the eigenvalues of the matrix $M$ are algebraic integers of $\mathbb{C}$-norm $p^{1/2} \neq 1$, the matrix $M - I$ is invertible, and we may solve the system to obtain the integrals $\int_{P'} Q \omega_i$. 

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$$\int_{P'} Q' \omega_i = \int_{\phi(P')} \phi(Q') \omega_i$$

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\int_{Q'} \omega_i = \int_{Q'} \phi^* \omega_i
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Integrals via Teichmüller, continued

- The linear system gives us the integral between different residue disks.

- Putting it all together, we have

\[
\int_P^Q \omega_i = \int_P^{P'} \omega_i + \int_P^{Q'} \omega_i + \int_Q^Q \omega_i
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\[ \int_{P}^{Q} \omega_{i} = \int_{P'}^{P} \omega_{i} + \int_{P'}^{Q'} \omega_{i} + \int_{Q'}^{Q} \omega_{i} \]
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Why \( p \)-adic heights?

Interpolating special values of the \( L \)-function in the usual BSD conjecture, we obtain a \( p \)-adic BSD conjecture:

**Conjecture (Mazur, Tate, and Teitelbaum)**

Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and let \( p \) be a prime of good ordinary reduction for \( E \). Then the algebraic rank of \( E \) equals \( \text{ord}_T(\mathcal{L}_p(E, T)) \) and

\[
\mathcal{L}_p^*(E, 0) = (1 - \alpha^{-1})^2 \cdot \frac{\prod \ell c_\ell \cdot |\text{III}(E/\mathbb{Q})| \cdot \text{Reg}_p(E)}{|E(\mathbb{Q})_{\text{tors}}|^2},
\]

with \( \alpha \) the unit root of \( x^2 - a_px + p \) and where the \( p \)-adic regulator \( \text{Reg}_p(E) \in \mathbb{Q}_p \) is defined analogously to the classical regulator:

- Take \( P_1, \ldots, P_r \) be a basis for \( E(\mathbb{Q}) \) modulo torsion.
- Then \( \text{Reg}_p(E) = \det(M) \), where \( M_{ij} = \langle P_i, P_j \rangle_p \), with a \( p \)-adic height pairing

\[
\langle \cdot, \cdot \rangle_p : E(\mathbb{Q}) \times E(\mathbb{Q}) \longrightarrow \mathbb{Q}_p.
\]
Notation

- **C**: genus $g$ hyperelliptic curve of the form $y^2 = f(x)$, with $\deg f(x) = 2g + 1$
- **K**: number field
- **k**: local field (char 0) with valuation ring $\mathcal{O}$, uniformizer $\pi$
- **F**: residue field, $\mathcal{O}/\pi\mathcal{O}$, with $|F| = q$.
- **J**: Jacobian of $C$ over $k$

We’ll assume $C$ has good ordinary reduction at $\pi$. 
The Coleman-Gross height pairing is a symmetric bilinear pairing

\[ h : \text{Div}^0(C) \times \text{Div}^0(C) \rightarrow \mathbb{Q}_p, \]

which can be written as a sum of local height pairings

\[ h = \sum_v h_v \]

over all finite places \( v \) of \( K \).

Techniques to compute \( h_v \) depend on \( \text{char}(F) \):

- \( \text{char}(F) \neq p \): intersection theory, as in Ph.D. thesis of Müller (’10)
- \( \text{char}(F) = p \): logarithms, normalized differentials, Coleman integration
Local height: residue characteristic $p$

Suppose $\text{char}(F) = p$, and that $k = K_v$, the completion at $v|p$.

**Definition**

Let $D_1, D_2 \in \text{Div}^0(C)$ have disjoint support and $\omega_{D_1}$ a normalized differential associated to $D_1$. The local height pairing at $v$ above $p$ is given by

\[ h_v(D_1, D_2) = \text{tr}_{k/Q_p} \left( \int_{D_2} \omega_{D_1} \right). \]
Local height: residue characteristic $p$

Constructing and using $\omega_{D_1}$: subtle, requires more cohomology

In brief:

- a map $\Psi$ that plays the role of a logarithm and aids in splitting $H^1_{dR}(C)$ (which gives us $\omega_{D_1}$)
- new tricks to compute Coleman integrals of differentials with poles in non-Weierstrass discs
Conjecture

Let $A_f$ be a modular abelian variety of dimension 2 attached to a newform $f$ and $p$ a prime of good ordinary reduction for $A_f$. Let $K_f$ be the real quadratic field containing the Hecke eigenvalue $a_p$. The Mordell-Weil rank of $A_f$ equals $\text{ord}_T(\mathcal{L}_p(A_f, T))$ and

$$\mathcal{L}_p^*(A_f, 0) \doteq N(K_f \otimes \mathbb{Q}_p)/\mathbb{Q}_p((1 - \alpha^{-1})^2) \cdot \frac{|\text{III}(A_f)| \cdot \text{Reg}_p(A_f) \cdot \prod \ell | N c_\ell}{|A_f(\mathbb{Q})_{\text{tors}}||A_f^\vee(\mathbb{Q})_{\text{tors}}|(\log(1 + p))^{r'}}$$

where $\mathcal{L}_p^*(A_f, 0)$ is the leading coefficient of the $p$-adic $L$-series $\mathcal{L}_p(A_f, T)$, $\alpha$ is the unit root, and $\doteq$ is an equality up to a sign $c = \pm 1$. 
Some candidate modular abelian varieties

- From “Empirical evidence for the Birch and Swinnerton-Dyer conjectures for modular Jacobians of genus 2 curves” (Flynn et al. ’01), a table of the associated curves and levels
- For each curve, the associated abelian variety has Mordell-Weil rank 2.

<table>
<thead>
<tr>
<th>N</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>67</td>
<td>$y^2 + (x^3 + x + 1)y = x^5 - x$</td>
</tr>
<tr>
<td>73</td>
<td>$y^2 + (x^3 + x^2 + 1)y = -x^5 - 2x^3 + x$</td>
</tr>
<tr>
<td>85</td>
<td>$y^2 + (x^3 + x^2 + x)y = x^4 + x^3 + 3x^2 - 2x + 1$</td>
</tr>
<tr>
<td>93</td>
<td>$y^2 + (x^3 + x^2 + 1)y = -2x^5 + x^4 + x^3$</td>
</tr>
<tr>
<td>103</td>
<td>$y^2 + (x^3 + x^2 + 1)y = x^5 + x^4$</td>
</tr>
<tr>
<td>107</td>
<td>$y^2 + (x^3 + x^2 + 1)y = x^4 - x^2 - x - 1$</td>
</tr>
<tr>
<td>115</td>
<td>$y^2 + (x^3 + x + 1)y = 2x^3 + x^2 + x$</td>
</tr>
<tr>
<td>125,A</td>
<td>$y^2 + (x^3 + x + 1)y = x^5 + 2x^4 + 2x^3 + x^2 - x - 1$</td>
</tr>
<tr>
<td>133,A</td>
<td>$y^2 + (x^3 + x^2 + 1)y = -x^5 + x^4 - 2x^3 + 2x^2 - 2x$</td>
</tr>
<tr>
<td>147</td>
<td>$y^2 + (x^3 + x^2 + x)y = x^5 + 2x^4 + x^3 + x^2 + 1$</td>
</tr>
<tr>
<td>161</td>
<td>$y^2 + (x^3 + x + 1)y = x^3 + 4x^2 + 4x + 1$</td>
</tr>
<tr>
<td>165</td>
<td>$y^2 + (x^3 + x^2 + x)y = x^5 + 2x^4 + 3x^3 + x^2 - 3x$</td>
</tr>
<tr>
<td>167</td>
<td>$y^2 + (x^3 + x + 1)y = -x^5 - x^3 - x^2 - 1$</td>
</tr>
<tr>
<td>177</td>
<td>$y^2 + (x^3 + x^2 + 1)y = x^5 + x^4 + x^3$</td>
</tr>
<tr>
<td>188</td>
<td>$y^2 = x^5 - x^4 + x^3 + x^2 - 2x + 1$</td>
</tr>
<tr>
<td>191</td>
<td>$y^2 + (x^3 + x + 1)y = -x^3 + x^2 + x$</td>
</tr>
</tbody>
</table>
To numerically verify $p$-adic BSD, need to compute $p$-adic regulators and $p$-adic special values.

For example, for $N = 188$, we have values of the $p$-adic regulator for various $p$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$p$-adic regulator</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$6 \cdot 7^2 + 5 \cdot 7^3 + 7^4 + 7^5 + 2 \cdot 7^6 + 6 \cdot 7^7 + O(7^8)$</td>
</tr>
<tr>
<td>11</td>
<td>$3 + 8 \cdot 11^2 + 6 \cdot 11^3 + 2 \cdot 11^4 + 11^5 + 10 \cdot 11^6 + O(11^7)$</td>
</tr>
<tr>
<td>13</td>
<td>$2 \cdot 13^2 + 3 \cdot 13^3 + 4 \cdot 13^4 + 9 \cdot 13^5 + 5 \cdot 13^6 + 10 \cdot 137 + O(13^8)$</td>
</tr>
<tr>
<td>17</td>
<td>$2 \cdot 17^2 + 13 \cdot 17^3 + 2 \cdot 17^4 + 4 \cdot 17^5 + 11 \cdot 17^7 + O(17^8)$</td>
</tr>
<tr>
<td>19</td>
<td>$12 \cdot 19^2 + 5 \cdot 19^3 + 10 \cdot 19^4 + 2 \cdot 19^5 + 9 \cdot 19^6 + 7 \cdot 19^7 + O(19^8)$</td>
</tr>
<tr>
<td>23</td>
<td>$19 \cdot 23^2 + 14 \cdot 23^3 + 2 \cdot 23^4 + 19 \cdot 23^5 + 20 \cdot 23^6 + 13 \cdot 23^7 + O(23^8)$</td>
</tr>
<tr>
<td>37</td>
<td>$33 \cdot 37^2 + 8 \cdot 37^3 + 28 \cdot 37^4 + 36 \cdot 37^5 + 2 \cdot 37^6 + 15 \cdot 37^7 + O(37^8)$</td>
</tr>
<tr>
<td>41</td>
<td>$6 \cdot 41^2 + 15 \cdot 41^3 + 7 \cdot 41^4 + 40 \cdot 41^6 + 5 \cdot 41^7 + O(41^8)$</td>
</tr>
<tr>
<td>43</td>
<td>$19 \cdot 43^2 + 29 \cdot 43^3 + 9 \cdot 43^5 + 2 \cdot 43^6 + 12 \cdot 43^7 + O(43^8)$</td>
</tr>
<tr>
<td>53</td>
<td>$46 \cdot 53^2 + 26 \cdot 53^3 + 19 \cdot 53^4 + 33 \cdot 53^5 + 17 \cdot 53^6 + 11 \cdot 53^7 + O(53^8)$</td>
</tr>
<tr>
<td>59</td>
<td>$51 \cdot 59^2 + 54 \cdot 59^3 + 37 \cdot 59^4 + 5 \cdot 59^5 + 58 \cdot 59^6 + 27 \cdot 59^7 + O(59^8)$</td>
</tr>
<tr>
<td>61</td>
<td>$37 \cdot 61^2 + 20 \cdot 61^3 + 37 \cdot 61^4 + 56 \cdot 61^5 + 32 \cdot 61^6 + 8 \cdot 61^7 + O(61^8)$</td>
</tr>
<tr>
<td>67</td>
<td>$22 \cdot 67^2 + 3 \cdot 67^3 + 26 \cdot 67^4 + 23 \cdot 67^5 + 11 \cdot 67^6 + 17 \cdot 67^7 + O(67^8)$</td>
</tr>
<tr>
<td>71</td>
<td>$68 \cdot 71^2 + 25 \cdot 71^3 + 62 \cdot 71^4 + 36 \cdot 71^5 + 62 \cdot 71^6 + 2 \cdot 71^7 + O(71^8)$</td>
</tr>
<tr>
<td>73</td>
<td>$60 \cdot 73^2 + 7 \cdot 73^3 + 12 \cdot 73^4 + 22 \cdot 73^5 + 46 \cdot 73^6 + 30 \cdot 73^7 + O(73^8)$</td>
</tr>
<tr>
<td>79</td>
<td>$21 \cdot 79^2 + 15 \cdot 79^3 + 17 \cdot 79^4 + 78 \cdot 79^5 + 2 \cdot 79^6 + 25 \cdot 79^7 + O(79^8)$</td>
</tr>
<tr>
<td>83</td>
<td>$44 \cdot 83^2 + 71 \cdot 83^3 + 32 \cdot 83^4 + 54 \cdot 83^5 + 18 \cdot 83^6 + 56 \cdot 83^7 + O(83^8)$</td>
</tr>
<tr>
<td>89</td>
<td>$30 \cdot 89^2 + 27 \cdot 89^3 + 11 \cdot 89^4 + 43 \cdot 89^5 + 27 \cdot 89^6 + 43 \cdot 89^7 + O(89^8)$</td>
</tr>
<tr>
<td>97</td>
<td>$50 \cdot 97^2 + 8 \cdot 97^3 + 77 \cdot 97^4 + 73 \cdot 97^5 + 82 \cdot 97^6 + 16 \cdot 97^7 + O(97^8)$</td>
</tr>
</tbody>
</table>
Here are the corresponding special values of the $p$-adic $L$-series:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$p$-adic special value</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$3 + 5 \cdot 7 + 4 \cdot 7^2 + 2 \cdot 7^4 + O(7^4)$</td>
</tr>
<tr>
<td>11</td>
<td>$5 + 10 \cdot 11 + 9 \cdot 11^2 + 5 \cdot 11^3 + O(11^4)$</td>
</tr>
<tr>
<td>13</td>
<td>$8 + 13 + 13^2 + 6 \cdot 13^3 + O(13^4)$</td>
</tr>
<tr>
<td>17</td>
<td>$8 + 15 \cdot 17 + 5 \cdot 17^2 + 2 \cdot 17^3 + O(17^4)$</td>
</tr>
<tr>
<td>19</td>
<td>$8 + 13 \cdot 19 + 6 \cdot 19^2 + 17 \cdot 19^3 + O(19^4)$</td>
</tr>
<tr>
<td>23</td>
<td>$5 + 15 \cdot 23 + 2 \cdot 23^2 + 8 \cdot 23^3 + O(23^4)$</td>
</tr>
<tr>
<td>37</td>
<td>$33 + 13 \cdot 37 + 22 \cdot 37^2 + O(37^3)$</td>
</tr>
<tr>
<td>41</td>
<td>$35 + 2 \cdot 41 + 37 \cdot 41^2 + O(41^3)$</td>
</tr>
<tr>
<td>43</td>
<td>$26 + 15 \cdot 43 + 19 \cdot 43^2 + O(43^3)$</td>
</tr>
<tr>
<td>53</td>
<td>$9 + 41 \cdot 53 + 13 \cdot 53^2 + O(53^3)$</td>
</tr>
<tr>
<td>59</td>
<td>$46 + 41 \cdot 59 + 2 \cdot 59^2 + O(59^3)$</td>
</tr>
<tr>
<td>61</td>
<td>$59 + 59 \cdot 61 + O(61^2)$</td>
</tr>
<tr>
<td>67</td>
<td>$37 + 67 + O(67^2)$</td>
</tr>
<tr>
<td>71</td>
<td>$66 + 14 \cdot 71 + O(71^2)$</td>
</tr>
<tr>
<td>73</td>
<td>$13 + 10 \cdot 73 + O(73^2)$</td>
</tr>
<tr>
<td>79</td>
<td>$26 + 13 \cdot 79 + O(79^2)$</td>
</tr>
<tr>
<td>83</td>
<td>$40 + 43 \cdot 83 + O(83^2)$</td>
</tr>
<tr>
<td>89</td>
<td>$24 + 54 \cdot 89 + O(89^2)$</td>
</tr>
<tr>
<td>97</td>
<td>$4 + O(97^2)$</td>
</tr>
</tbody>
</table>
Future directions

- Heights after Harvey
  - Our basic algorithm has linear runtime dependence on the prime $p$, arising from the corresponding dependence in Kedlaya’s algorithm; could possibly follow Harvey’s variant of Kedlaya’s algorithm to reduce this to square-root dependence on $p$

- Beyond hyperelliptic curves
  - Convert algorithms for computing Frobenius actions on de Rham cohomology (Gaudry-Gürel, Castryck-Denef-Vercauteren) into algorithms for computing Coleman integrals on such curves
  - Use Abbott-Kedlaya-Roe (and Harvey’s recent version) for smooth hypersurfaces in projective space (which can be generalized to nondegenerate hypersurfaces in toric varieties) to carry out Coleman integration on these surfaces