Habits of Mind: An organizing principle for mathematics curriculum and instruction

Michelle Manes
Department of Mathematics
Brown University
mmanes@math.brown.edu
• Position paper by Al Cuoco, E. Paul Goldenberg, and June Mark.

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  Search for the title “Habits of Mind: An Organizing Principle for Mathematics Curricula.”
Analytic geometry or fractal geometry?
Analytic geometry or fractal geometry?

Modeling with algebra or modeling with spreadsheets?
Analytic geometry or fractal geometry?

Modeling with algebra or modeling with spreadsheets?

Graph theory or solid geometry?
Analytic geometry or fractal geometry?

Modeling with algebra or modeling with spreadsheets?

Graph theory or solid geometry?

These are the wrong questions.
Traditional courses

- mechanisms for communicating established results
Traditional courses

• mechanisms for communicating established results
• give students a bag of facts
Traditional courses

• mechanisms for communicating established results
• give students a bag of facts
• reform means replacing one set of established results by another
Traditional courses

• mechanisms for communicating established results
• give students a bag of facts
• reform means replacing one set of established results by another
• learn properties, apply the properties, move on
New view of curriculum

Results of mathematics research

Methods used to create mathematics
New view of curriculum

Results of mathematics research

Methods used to create mathematics
New view of curriculum

Results of mathematics research  Methods used to create mathematics
Goals

• Train large numbers of university mathematicians.
Goals

- Train large numbers of university mathematicians.
Goals

• Help students learn and adopt some of the ways that mathematicians think about problems.
Goals

• Help students learn and adopt some of the ways that mathematicians *think* about problems.

• Let students in on the process of creating, inventing, conjecturing, and experimenting.
Curricula should encourage

- false starts
Curricula should encourage

- false starts
- experiments
Curricula should encourage

- false starts
- experiments
- calculations
Curricula should encourage

- false starts
- experiments
- calculations
- special cases
Curricula should encourage

- false starts
- experiments
- calculations
- special cases
- using lemmas
Curricula should encourage:

- false starts
- experiments
- calculations
- special cases
- using lemmas
- looking for logical connections
A caveat

Students think about mathematics the way mathematicians do.
A caveat

Students think about mathematics the way mathematicians do.

NOT

Students think about the same topics that mathematicians do.
What students say about their mathematics courses

• It’s about triangles.
• It’s about solving equations.
• It’s about doing percent.
What we want students to say about mathematics

It’s about ways of solving problems.
What we want students to say about mathematics

It’s about ways of solving problems.

(Not: It’s about the five steps for solving a problem!)
Students should be pattern sniffers.
Below is a copy of the diagram you made in the Explore. The word “Start” has been replaced by the numeral 1, and the rows have been labeled.

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There are many patterns in this triangle. For example, each row reads the same forward as it does backward.

1. Describe as many patterns in the triangle as you can.

2. To add more rows to the triangle, you could count paths as you did in the Explore—but that might take a lot of time. Instead, use some of the patterns you found in Problem 1 to extend the triangle to Row 7. You may not be able to figure out all the numbers, but fill in as many as you can.
Students should be experimenters.
The Airport Problem  Where should you put point $D$ if you want to make $DA + DB + DC$ as small as possible?

Should it go here?

Here?

Here?

Here?

Or maybe even here?
Idea 1: Look at Simpler Problems and Special Cases

4. What if there are only two cities? Where can the airport go?
   
   - City B
   
   City A •

5. What if there are three cities, but they do *not* form a triangle? That is, what if they are collinear?

   a. Make an argument for an airport site that minimizes the total distance.
   
   b. Can you find a *fair* solution—a spot that is equally distant to each city?

   - City C
   
   City A •
What if one city is very far away from the other two? The sketch below doesn't necessarily show the best position for the airport. Where (roughly) would that position be? Explain your reasoning. How would the best position change if City C were twice as far away from $AB$? This may be a good question to investigate using geometry software.

What if the cities form an equilateral triangle? The center of the triangle looks like a good spot for the airport. Test enough locations to see if the center spot really minimizes the total distance when the cities are equidistant from each other.

Here the cities form an equilateral triangle.
Students should be describers.
Give precise directions
Give precise directions
Invent notation
Invent notation
Argue
Zoe noticed that the square pattern and the double-staircase pattern grow the same way numerically.

1 block  4 blocks  9 blocks
Argue

I was surprised at first because they look so different. But then I found a way to make the square from the staircase!

Say you start with this staircase.

Leave the base (the bottom layer) alone. Move the top part (which is really the staircase one stage smaller) forward so it sits inside the base.

Repeat this until the last block fills in the square!
Students should be tinkerers.
Students should be tinkerers.

“What is $3^5i$?”
Students should be inventors.

\[a \diamond b = \frac{a + 2b}{3}\]
Students should be inventors.

\[ a \diamond b = \frac{a + 2b}{3} \]

Alice offers to sell Bob her iPod for $100. Bob offers $50. Alice comes down to $75, to which Bob offers $62.50. They continue haggling like this. How much will Bob pay for the iPod?
Students should look for isomorphic structures.
1.1 Four Problems to Consider

Try the following combinatorics problems:

1. How many three-digit numbers can you make using only the digits 1 and 2? (Of course, in each number you may use a digit more than once.)

2. In a kindergarten class, children are asked to color each of 3 different shapes either green or red. Following these directions, how many different colorings are possible?

3. A pizzeria has three choices of toppings: onions, mushrooms, and pepperoni. You check off on the order form what toppings you want, if any. How many different sorts of combination pizzas are possible?

4. A coin is tossed three times. One of the possible outcomes is tail-head-head, another one is head-tail-head. How many possible outcomes are there?
Students should be visualizers.
Students should be visualizers.

How many windows are in your house?
Categories of visualization

• reasoning about subsets of 2D or 3D space
• visualizing data
• visualizing relationships
• visualizing processes
• reasoning by continuity
• visualizing calculations
Students should be conjecturers.
Inspi

to inspi :side :angle :increment
   forward :side
   right :angle
   inspi :side (:angle + :increment) :increment
end
Two incorrect conjectures

- If $\angle + \text{increment} = 6$, there are two pods
Two incorrect conjectures

• If \( \text{angle} + \text{increment} = 6 \), there are two pods

• If \( \text{increment} = 1 \), there are two pods
Students should be guessers.
Students should be guessers.

“Guess x.”
Mathematicians talk big and think small.

Let $K$ be a field, $V$ and $W$ vector spaces over $K$ of dimensions $n$ and $m$ respectively. If $n > m$ and $T: V \to W$ is linear, then $T$ is not one-to-one.
Mathematicians talk big and think small.

Let $K$ be a field, $V$ and $W$ vector spaces over $K$ of dimensions $n$ and $m$ respectively. If $n > m$ and $T: V \rightarrow W$ is linear, then $T$ is not one-to-one.

“You can’t map three dimensions into two with a matrix unless things get scrunched.”
Mathematicians talk small and think big.
Mathematicians talk small and think big.

Ever notice that a sum of two squares times a sum of two squares is also a sum of two squares?

\[
\begin{align*}
13 &= 9 + 4 \\
5 &= 4 + 1 \\
13 \times 5 &= 65 = 16 + 49
\end{align*}
\]
Mathematicians talk small and think big.

This can be explained with Gaussian integers.

\[
\begin{align*}
13 &= (3 + 2i)(3 - 2i) \\
5 &= (2 + i)(2 - i) \\
13 \times 5 &= (3 + 2i)(3 - 2i) \times (2 + i)(2 - i) \\
&= (3 + 2i)(2 + i) \times (3 - 2i)(2 - i) \\
&= (4 + 7i)(4 - 7i) \\
&= 16 + 49
\end{align*}
\]
Mathematicians mix deduction and experiment.

• Experimental evidence is not enough.
Are there integer solutions to this equation?

\[ x^3 + y^3 + z^3 = 30 \]
Are there integer solutions to this equation?

\[ x^3 + y^3 + z^3 = 30 \]

Yes, but you probably won’t find them experimentally.

\((-283059965, -221888517, 2220422932)\)
Mathematicians mix deduction and experiment.

- Experimental evidence is not enough.
- The proof of a statement suggests new theorems.
Mathematicians mix deduction and experiment.

- Experimental evidence is not enough.
- The proof of a statement suggests new theorems.
- Proof is what sets mathematics apart from other disciplines. In a sense, it is the mathematical habit of mind.