POWER SUMS, BERNOULLI NUMBERS, AND RIEMANN'S ζ-FUNCTION.

1. Power sums

We begin with a definition of power sums, $S_m(n)$. This quantity is defined for positive integers $m > 0$ and $n > 1$ as the sum of m-th powers of the first $n-1$ integers:

$$S_m(n) = \sum_{i=1}^{n-1} i^m = 1^m + \ldots + (n-1)^m.$$

Our target is to find an explicite closed formula for $S_m(n)$ in terms of m and n. We already know some special cases of the formula in question. In particular, it is easy to see that

$$S_m(2) = 1,$$

and it is not difficult to verify the following formulas using mathematical induction

$$S_1(n) = \frac{n(n-1)}{2},$$

$$S_2(n) = \frac{n(n-1)(2n-1)}{6},$$

$$S_3(n) = \frac{n^2(n-1)^2}{4}.$$

Although it is not difficult to prove the above formulas if they are stated, it does not seem easy to guess the exact shape of them. One can, however, make a general guess, which we formulate and prove as a proposition.

Proposition 1. *$S_m(n)$ may be represented as a polynomial in n of degree $m+1$. The constant term of this polynomial is zero, and the leading term of this polynomial is $n^{m+1}/(m+1)$*

Proof. Recall the standard binomial formula: for a positive integer d, one has

$$(l+k)^d = l^d + \binom{d}{1} l^{d-1}k + \binom{d}{2} l^{d-2}k^2 + \ldots + k^d.$$

We will need a special case of this formula. Namely, we put $d = m+1,\ \ l = 1$, and substruct the term k^{m+1} from both sides to obtain

$$(1+k)^{m+1} - k^{m+1} = 1 + \binom{m+1}{1} k + \binom{m+1}{2} k^2 + \ldots + \binom{m+1}{m} k^m.$$

Let us now plug in the values $k = 0, 1, 2, \ldots, n-1$ into the latter identity, and write down the n formulas obtained in this way:

$$\begin{array}{rclllll}
1^{m+1}-0^{m+1} & = & 1 & & & & \\
2^{m+1}-1^{m+1} & = & 1+ & \binom{m+1}{1}1+ & \binom{m+1}{2}1^2+ & \ldots+ & \binom{m+1}{m}1^m \\
3^{m+1}-2^{m+1} & = & 1+ & \binom{m+1}{1}2+ & \binom{m+1}{2}2^2+ & \ldots+ & \binom{m+1}{m}2^m \\
\ldots & \ldots & \ldots & & & & \\
\ldots & \ldots & \ldots & & & & \\
\ldots & \ldots & \ldots & & & & \\
n^{m+1}-(n-1)^{m+1} & = & 1+ & \binom{m+1}{1}(n-1)+ & \binom{m+1}{2}(n-1)^2+ & \ldots+ & \binom{m+1}{m}(n-1)^m.
\end{array}$$

Let us now sum these n identities together, observing the cancellations in the left-hand side and the appearence of power sums in the right:

$$n^{m+1} = n + \binom{m+1}{1}S_1(n) + \binom{m+1}{2}S_2(n) + \ldots + \binom{m+1}{m}S_m(n).$$

Taking into the account that

$$\binom{m+1}{m} = m+1$$

we obtain finally

$$(m+1)S_m(n) = n^{m+1} - \binom{m+1}{1}S_1(n) - \binom{m+1}{2}S_2(n) - \ldots - \binom{m+1}{m-1}S_{m-1}(n)$$

Exercise 1. *Finish the proof of Proposition 1 using the latter identity and a mathematical induction argument.*

□

Exercise 2. *Use the proof of Proposition 1 in order to find the formula for $S_5(n)$.*

2. Bernoulli numbers

Althuogh Proposition 1 provides valuable qualitative information about the formulas for $S_m(n)$, and the proof of this proposition actually allows to find these formulas, we will now present another way to write these formulas down. In order to do so we define the sequence $B_0, B_1, B_2 \ldots$ of Bernoulli numbers as the coefficients of the Maclaurin series for the function $t/(e^t-1)$:

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

One can reformulate the above formula by saying that the function $t/(e^t-1)$ is the *generating function* for the sequence of rational numbers $B_m/m!$. A straightforward but tedious calculation of the Maclaurin series allows us to find the first few Bernoulli numbers:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0.\ B_6 = \frac{1}{42},\ \ldots$$

We now make Proposition 1 more precise by establishing a connection between power sums and Bernoulli numbers.

Theorem. (Bernoulli) *For an integers $n > 1$ and $m > 0$*

$$(m+1)S_m(n) = \sum_{j=0}^{m} \binom{m+1}{j} B_j n^{m+1-j}.$$

Proof. We begin with a standard Maclaurin series for the exponential function:

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}.$$

We put $x = kt$ in the above formula, an obtain

$$e^{kt} = \sum_{m=0}^{\infty} k^m \frac{t^m}{m!}.$$

We now write the latter formula down for $k = 0, 1, 2, \ldots, n-1$:

$$\begin{aligned}
1 &= 1 \\
e^{1t} &= \sum_{m=0}^{\infty} 1^m \frac{t^m}{m!} \\
e^{2t} &= \sum_{m=0}^{\infty} 2^m \frac{t^m}{m!} \\
\ldots &= \ldots \\
\ldots &= \ldots \\
\ldots &= \ldots \\
e^{(n-1)t} &= \sum_{m=0}^{\infty} (n-1)^m \frac{t^m}{m!}.
\end{aligned}$$

We add these n identities, and observe the appearance of the power sums in the right-hand side:

$$1 + e^{1t} + e^{2t} + \ldots + e^{(n-1)t} = 1 + \sum_{m=0}^{\infty} S_m(n) \frac{t^m}{m!} \tag{1}$$

Transform the left-hand side using the geometric series formula

$$1 + e^{1t} + e^{2t} + \ldots + e^{(n-1)t} = \frac{e^{nt}-1}{e^t-1} = \frac{e^{nt}-1}{t} \frac{t}{e^t-1},$$

notice that both factors have Maclaurin series

$$\frac{e^{nt}-1}{t} = \sum_{k=1}^{\infty} n^k \frac{t^{k-1}}{k!} \qquad \frac{t}{e^t-1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!},$$

and multiply these two series

$$\frac{e^{nt}-1}{t} \frac{t}{e^t-1} = \sum_{m=0}^{\infty} \sum_{\substack{m=k+j-1 \\ k\geq 1 \quad j\geq 0}} n^k B_j \frac{t^m}{k!j!} = 1 + \sum_{m=0}^{\infty} S_m(n) \frac{t^m}{m!}.$$

Equate like powers of t in the above identity and obtain

$$\frac{S_m(n)}{m!} = \sum_{\substack{m=k+j-1 \\ k\geq 1 \quad j\geq 0}} n^k B_j \frac{1}{k!j!}.$$

In order to finish the proof we multiply both sides of this identity by $(m+1)!$, and use the obvious fact that

$$\frac{(m+1)!}{(m+1-j)!j!} = \binom{m+1}{j}.$$

□

Exercise 3. *Prove that $B_{2n+1} = 0$ for $n \geq 1$.*
Hint. The function $t/(e^t - 1) + t/2$ is even.

Exercise 4. *Define the sequence of rational numbers $b_0, b_1, b_2, \ldots$ as follows. Put $b_0 = 0$, and for $m \geq 1$*

$$(m+1)b_m = -\sum_{k=0}^{m-1} \binom{m+1}{k} b_k.$$

Prove that $b_m = B_m$.
Hint. In the definition of Bernoulli numbers, multiply both sides by $e^t - 1$, and write the Maclourin series in t for this function. Equate like coefficients of like powers of t, and show that Bernoulli numbers satisfy the above identity. Explain, why this fact implies $b_m = B_m$.
Show that it suffices to prove that Bernoulli numbers B_m satisfy the above identity.

This exercise presents an alternative definition for Bernoulli numbers.

3. Riemann's ζ-function

Riemann's ζ-function is a function on s defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Honestly, this series is considered for a complex variable s. We do not do that here. Instead, for a positive integer m, we consider the values

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{1}{1^{2m}} + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \ldots\ .$$

This series obviously converges (why?), and may be taken as an analogue of the power sums. The sums, however are now infinite, and the exponents are negative, thus the analogy is not close. This makes even more impressive the following classical result which connects these infinite sums to Bernoulli numbers

Theorem. (Euler)

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

Proof. The proof requires the following identity from real analysis

$$\cot x = \frac{1}{x} - 2\sum_{n=1}^{\infty} \frac{x}{n^2\pi^2 - x^2}.$$

We take this identity for granted.

We will now make use of the *infinite* geometric series formula in order to transform the summands in the right-hand side:

$$\frac{x}{n^2\pi^2 - x^2} = \frac{x}{n^2\pi^2}\frac{1}{1-(x/n\pi)^2} = \frac{x}{n^2\pi^2}\sum_{l=0}^{\infty}\left(\frac{x}{n\pi}\right)^{2l}.$$

We thus obtain

$$x\cot x = 1 - 2\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{x}{n\pi}\right)^{2l} = 1 - 2\sum_{l=1}^{\infty}\frac{x^{2l}}{\pi^{2l}}\sum_{n=1}^{\infty}\frac{1}{n^{2l}} = 1 - 2\sum_{l=1}^{\infty}\frac{x^{2l}}{\pi^{2l}}\zeta(2l).$$

Now let us transform $x\cot x$. We need some basic facts from complex analysis, namely

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad , \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

which we also take for granted. Thus

$$x\cot x = ix\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = ix\frac{e^{2ix} + 1}{e^{2ix} - 1} = ix\frac{e^{2ix} - 1 + 2}{e^{2ix} - 1} = ix + \frac{2ix}{e^{2ix} - 1} = ix + \sum_{m=0}^{\infty} B_m\frac{(2ix)^m}{m!}$$

We thus have two power series (in x) representations for the function $x\cot x$, and can equate them. After that we equate like powers of x, and finish the proof of the theorem.

□