

1. PROBLEMS FROM 2007 CONTEST

PROBLEM 1A _____

Do there exist 10 natural numbers such that none one of them is divisible by another one, and the square of any one of them is divisible by any other of the original numbers?

PROBLEM 2A _____

Consider 101 natural numbers not exceeding 200. Prove that at least one of them is divisible by another one.

PROBLEM 3A _____

Given a triple of numbers one is allowed to perform the following operation. Take any two of them, say a and b out, and put $(a + b)/\sqrt{2}$ and $(a - b)/\sqrt{2}$ instead. Is it possible to produce the triple $(1, \sqrt{2}, 1 + \sqrt{2})$ out of $(2, \sqrt{2}, 1/\sqrt{2})$ after a number of iterations of the allowed operation?

PROBLEM 4A _____

The sequence $1, 11, 111, 1111, 11111, \dots$ does not contain numbers divisible by 2005 and 2006. Indeed, the former number is divisible by 5, and the latter number is divisible by 2, whereas neither member of the sequence is divisible by 2 or 5. Does this sequence contain any member divisible by 2007?

PROBLEM 5A _____

There are n points in the plane. All the midpoints of all segments which have given points as their endpoints are marked. Prove that at least $2n - 3$ points are marked.

PROBLEM 1B _____

A country has a hundred airports. All distances between the pairs of airports are different. One day planes starting in every airport head to the closest airport. Assume that one and only one plane flies from each airport. What is the maximal number of planes landing this day at the same airport?

PROBLEM 2B _____

Consider a hundred of integers. Prove that one can pick several of them (maybe only one) such that their sum is divisible by 100.

PROBLEM 3B _____

Several positive numbers are arranged into a rectangular array. The product of the sum of numbers in any column with the sum of the numbers in any row equals the number which occupies the intersection of this column and this row. Find the sum of all numbers in the array.

PROBLEM 4B _____

There are $2k + 1$ cards with the numbers $1, 2, 3, \dots, 2k + 1$ written on them. What is the maximal number of cards which one can pick such that no one chosen number equals the sum of two other chosen numbers?

PROBLEM 5B _____

A finite set of points is situated in the plane in such a way that every straight line through any two of them passes through at least one more point from this set. Prove that all the points belong to one straight line.

1. PROBLEMS FROM 2008 CONTEST

PROBLEM 1A _____

Is it possible to situate 2008 line segments on the plane such that every endpoint of every segment belongs to the interior of another segment?

PROBLEM 2A _____

Prove that it is possible to choose three out of every seven positive integers, such that their sum is divisible by 3.

PROBLEM 3A _____

There are 31 planets, and an astronomer residing on every one of them. All distances between the planets are pairwise distinct. Every astronomer is scrutinizing the planet which is the closest to his or her own. Prove that there is a planet such that no astronomer directs the telescope to it.

PROBLEM 4A _____

A square of size 1×1 is cut into a number of rectangles (the cutting lines are parallel to the sides of the square). Take the smallest dimension of every rectangle. Can their sum be smaller than 1?

PROBLEM 5A _____

There are 20 different positive integers not exceeding 69. Prove that there are four equal pairwise differences of these numbers.

PROBLEM 1B _____

May it happen that the product of two consecutive positive integers is equal to the product of two consecutive even numbers?

PROBLEM 2B _____

How many solutions in positive integers a , b and c does the equation $a^{15} + b^{15} = c^{16}$ have?

PROBLEM 3B _____

There are 200 of red and blue chips (a hundred of red and a hundred of blue) in a row in some order. Prove that there is an interval out of 10 chips which contains exactly 5 chips of each color.

PROBLEM 4B _____

A tourist begins her trip at 6am, goes uphill during the day, and spends the night in her tent on the top of the hill. Next day she begins her trip back also at 6am, takes the same trail downhill, and quickly reaches the initial point of the trip. Prove that there is a point on her trail such that she has passed this point at exactly the same time of the day in both days of the trip.

PROBLEM 5B _____

A square field of size 10×10 consists of a hundred of 1×1 square regions separated with fences. Some 9 of these regions are infected with certain weeds. Infected regions stay infected forever. Moreover, every year the weeds spread into a new region if it shares at least two sides with infected regions. Prove that the weeds will never occupy the whole field.

3. PROBLEMS FROM 2009 CONTEST

PROBLEM 1A _____

John and Jane play a game. They have a rectangular table and a big enough supply of various coins (pennies, nickels, dimes, and quarters). They take turns putting a coin on the table. The person who cannot make a move, because there is no more free space loses the game. Coins should neither overlap nor fall down from the table (the center of every coin must be inside the rectangle). Jane makes the first move. Prove that there is a strategy which guarantees her victory.

PROBLEM 2A _____

The 2008 numbers

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{2008}, \frac{1}{2009}$$

are written on a blackboard. One may erase two of them, say, a and b , and write the number $a + b + ab$ instead. Obviously, only one number is left on the blackboard after 2007 such moves. Prove that the last number remaining is always bigger than 1000 (no matter which order the numbers were erased).square

PROBLEM 3A _____

Let a_1, a_2, a_3, \dots be an infinite sequence of distinct integers. Assume that every integer in this sequence is bigger than one. Prove that this sequence must contain infinitely many members a_i such that $a_i > i$.

PROBLEM 4A _____

Let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

It is well-known that H_n may be arbitrarily large if n is big enough. Prove that H_n is never an integer for $n > 1$.

PROBLEM 5A _____

Is it possible to arrange integers from 1 to 81 into a square 9×9 table in a way such that the difference between any two numbers in neighbor cells does not exceed 5? Two cells are called neighbors if they share a side.

PROBLEM 1B _____

12 blacksmiths have to shoe 15 horses. It takes 5 minutes for one blacksmith to shoe one hoof of a horse. What is the minimum amount of time which the blacksmiths need for this job? Note that a horse needs to always keep at least three legs on the ground.

PROBLEM 2B _____

Do there exist 10 natural numbers such that neither one of them is divisible by another, yet the square of any one of them is divisible by each of the other 9 numbers?

PROBLEM 3B _____

Being situated at a point O a camera captures two subjects A and B if the angle AOB is smaller than 179° . There are 1000 such cameras on the plane that all take a picture at the exact same moment. Is it possible that every picture contains 999 cameras?

PROBLEM 4B _____

There is string $a_1, a_2, a_3, \dots, a_{2009}$ of 2009 real numbers. Prove that there is a substring (which may contain one or several numbers) such that the sum of the numbers in the substring is closer than 0.001 to an integer.

PROBLEM 5B _____

There was a sheet of grid paper with a size of 29×29 . Out of this sheet, ninety nine 2×2 square cells were cut out. Prove that it is still possible to cut out at least one more 2×2 square.

4. PROBLEMS FROM 2010 CONTEST

PROBLEM 1A _____

Divide the natural numbers from 1 to 100000 into two sets: the even numbers and the odd numbers. Consider the sum of all the digits of all the numbers in each set. Which sum is larger, and what is the difference between the sums?

PROBLEM 2A _____

A positive integer b is obtained from a positive integer a by permuting (rearranging) its digits. Can it happen that

$$a + b = \underbrace{999 \dots 99}_{999 \text{ digits of } 9} ?$$

PROBLEM 3A _____

Do there exist 100 positive integers such that their sum is equal to their least common multiple (i.e., the smallest positive integer that is divisible by all the 100 integers)?

PROBLEM 4A _____

A finite set of $n \geq 3$ points on the plane which do not lie on one straight line is given. Prove that there exists a circle which goes through 3 of them such that no point from the given set is inside this circle.

PROBLEM 5A _____

Your friend thought of a polynomial $p(x) = a_0 + a_1x + \dots a_nx^n$ of some positive integer degree n with non-negative integer coefficients a_n . Can you determine what this polynomial is (i.e. determine all its coefficients) if you ask the friend about the value of the polynomial at only two different points?

PROBLEM 1B _____

Do there exist 2010 different positive integers $c_1, c_2, \dots, c_{2010}$ such that the sum of squares of every two neighbors $c_i^2 + c_{i+1}^2$ is a perfect square (i.e., the square of some integer)?

PROBLEM 2B _____

A square is divided by 198 straight lines into ten thousand rectangles (99 lines are parallel to one side of the square, and 99 to the other). Out of these ten thousand rectangles exactly 99 are squares. Can it happen that these squares are all different sizes?

PROBLEM 3B _____

There are 47 points situated on the plane such that out of every three points there exist two with a distance between them smaller than 1. Can it happen that every circle of radius 1 contains no more than 23 points?

PROBLEM 4B _____

Does there exist a set of 53 different positive integers such that the sum of all of them does not exceed 2010, and the sum of any two of them is different from 53?

PROBLEM 5B _____

An interval of length one is covered by a finite number of intervals. (The number of covering intervals may be large, and they may intersect.) Is it always possible to find a subset of the covering intervals such that no two of them intersect, and the sum of their lengths is at least .5?

5. PROBLEMS FROM 2011 CONTEST

PROBLEM 1A _____

There are $N > 1$ towns in a country. All the distances between them are pairwise distinct. A traveler starts in town A, and chooses as the destination the town which is the most distant from A. After reaching his destination, he continues to follow the same strategy, every time choosing the most distant town as his next destination. Eventually the traveler returns back to A after visiting all N towns. Find all possible values of N .

PROBLEM 2A _____

Is it possible to deposit 50 checks of \$370, \$372, ..., \$468 into seven accounts so that every account receives no more than \$3000?

PROBLEM 3A _____

Can one represent the number 2010/2011 as a sum of inverses of integers that are all different?

Some numbers can be represented in this way: for example,

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{15}.$$

PROBLEM 4A _____

Do there exist two different positive integers $m \neq n$ such that the number 2^n can be obtained from the number 2^m by a permutation of its digits (in the decimal representation)?

PROBLEM 5A _____

Is it possible to cut a square into 2011 smaller squares? Each of the smaller squares can be of any size.

PROBLEM 1B _____

Is it possible to arrange the 16 integers $1, 2, 3, 4, \dots, 16$ in a circle such that the sum of every two neighbors is a square of an integer?

Note that it is relatively easy to find an arrangement of them in a line interval with the required property:

16, 9, 7, 2, 14, 11, 5, 4, 12, 13, 3, 6, 10, 15, 1, 8

PROBLEM 2B _____

Do there exist 2011 real numbers such that the sum of any seventeen of them is positive while the sum of all of them is negative?

PROBLEM 3B _____

In the sequence of integers

$1, 2, 4, 8, 16, 23, 28, 38, 49, 62, 70, \dots$

the first term is 1, and every next term is the sum of the previous one and the sum of the digits of the previous one. Does the number 123456789 eventually show up in this sequence?

PROBLEM 4B _____

A sequence

$1, 2, 3, 4, 0, 9, 6, 9, 4, 8, 7, 8, 7, 0, 2, \dots$

The first four terms of the sequence are 1, 2, 3, 4, and after that every term is the last digit in the sum of the previous four terms of the sequence.

Does 8, 1, 2, 3 ever show up in this sequence?

PROBLEM 5B _____

Do there exist 2011 integers not divisible by 2011 such that the sum of any several of them is also not divisible by 2011?

6. PROBLEMS FROM WINTER 2011 CONTEST

PROBLEM 1A _____

Beginning with

$$1, 2, 3, 4, \dots, 2222,$$

you can perform the following operation any number of times: pick any two elements, and add 1 to both of them. This way, is it possible to make all of the elements equal?

PROBLEM 2A _____

For the set

$$\{a_1, a_2, a_3, \dots, a_{2011}\},$$

for any odd positive integer n ,

$$a_1^n + a_2^n + a_3^n + \dots + a_{2011}^n = 0.$$

Note that the condition would obviously hold, for example, if

$$a_1 = -a_2, a_3 = -a_4, \dots, a_{2009} = -a_{2010}, a_{2011} = 0.$$

Is it possible for the condition to hold if none of the elements of the set are equal to 0?

PROBLEM 3A _____

Does there exist a power of 3 such that its last decimal digits are 001, that is

$$3^m = \dots\dots\dots001$$

for some positive integer m ?

PROBLEM 4A _____

You are given a rectangular 2011×2012 table, which consists of 2011×2012 numbers. Any number of times, you can pick a row or a column, and change the signs of all the numbers there. Is it always possible to obtain a table where for all rows columns, the sum of the numbers in the row or column is non-negative?

PROBLEM 5A _____

Is it possible to make a 7×7 square out of 16 rectangles of size 1×3 and one square of size 1×1 such that the 1×1 square is neither in the center of the big (7×7) square nor adjacent to the sides of the big square?

PROBLEM 1B _____

Find four integers a, b, c , and d such that both $a^2 + 2cd + b^2$ and $c^2 + 2ab + d^2$ are perfect squares (i.e. squares of integers).

PROBLEM 2B _____

Consider the three sets of integers

a)

1, 2, 3, ..., 2011

b)

1, 2, 3, ..., 2012

c)

1, 2, 3, ..., 2013

An arbitrary number of times you can exchange any two numbers from a set for their difference. This way, can you produce a set of zeros out of the given set?

PROBLEM 3B _____

Find three different non-zero integers a, b , and c such that $a + b + c = 0$, and $a^{13} + b^{13} + c^{13}$ is a square of an integer.

PROBLEM 4B _____

The currencies in the two neighboring countries of Den and Dun are called the Denar and the Dunar respectively. In Den, the exchange rate is 10 dunars for 1 denar, while in Dun they give 10 denars for 1 dunar. A financier with an initial capital of 1 dunar is allowed to cross the border freely, and to perform any exchange as many times as he wants. Is it possible for the financier to have equal quantities of denars and dunars on hand at some point?

PROBLEM 5B _____

There are 30 chess players in a round-robin tournament (every player plays every other one exactly once). Can it happen that, at some point, the numbers of games played by the players are all distinct?