EULER PRODUCTS.

Recall that Riemann’s ζ-function is a function on $s$ defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. $$

This series determines a function in complex $s$ as a complex variable. It converges for $\Re(s) > 1$, and admits analytic continuation. We do not need all these facts here.

We consider ζ-function as another type of generating function. Namely, instead of a function on $z$

$$A(z) = \sum_{n=0}^{\infty} a_n z^n,$$

one can consider a function on $s$

$$Z(s) = \sum_{n=1}^{\infty} a_n n^{-s}. $$

We will ignore all analytical questions related to the convergence of infinite series (in fact, the series converges in all cases under our consideration). The two functions are, of course, tightly connected, but we will not consider this connection here. Instead, we will concentrate on an interesting special case.

From now on we assume that $a_n = f(n)$ for a multiplicative arithmetic function $f$

$$Z_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}. $$

Riemann’s ζ-function shows up as a special case when $f(n) = 1$ for each $n$. It is easy to prove (do that as an exercise) the Euler product decomposition

$$Z_f(s) = \prod_p \left( \sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}} \right),$$

where the infinite product is taken over all primes $p$.

In the particular case when $f(n) = 1$ for each $n$ we obtain

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

because in this case

$$\sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}} = \sum_{m=0}^{\infty} \frac{1}{p^{ms}} = \frac{1}{1 - p^{-s}}$$

by the geometric series formula.

Our goal here is to find simple relations between Riemann’s ζ-function and functions $Z_f(s)$ for various arithmetic functions $f$. 
Theorem 1. The following identities hold.

\[ \sum_{n=1}^{\infty} \mu(n)n^{-s} = \frac{1}{\zeta(s)} \]

\[ \sum_{n=1}^{\infty} \sigma(n)n^{-s} = \zeta(s)\zeta(s - 1) \]

\[ \sum_{n=1}^{\infty} \phi(n)n^{-s} = \frac{\zeta(s - 1)}{\zeta(s)} \]

\[ \sum_{n=1}^{\infty} d(n)n^{-s} = \zeta(s)^2 \]

Proof. The proofs of these identities are similar, and we will prove here only the last one. The proofs of the first three identities are left as exercises.

We begin with rewriting (1) for \( f(n) = d(n) \) as

\[ \sum_{n=1}^{\infty} d(n) = \prod_{p} \left( \sum_{m=0}^{\infty} \frac{d(p^m)}{p^{ms}} \right) = \prod_{p} \left( \sum_{m=0}^{\infty} \frac{m+1}{p^{ms}} \right), \]

because \( d(p^m) = m+1 \).

In order to evaluate the infinite sums we make use of the geometric series formula

\[ \sum_{m=0}^{\infty} q^m = \frac{1}{1-q} \]

and its derivative with respect to \( q \)

\[ \sum_{m=0}^{\infty} mq^{m-1} = \frac{1}{(1-q)^2}. \]

We obtain that

\[ \sum_{m=0}^{\infty} \frac{m+1}{p^{ms}} = \sum_{m=0}^{\infty} \frac{m}{p^{ms}} + \sum_{m=0}^{\infty} \frac{1}{p^{ms}} = \frac{p^{-s}}{(1-p^{-s})^2} + \frac{1}{1-p^{-s}} = \frac{1}{(1-p^{-s})^2} \]

Thus

\[ \sum_{n=1}^{\infty} d(n) = \prod_{p} \frac{1}{(1-p^{-s})^2} = \zeta(s)^2 \]

by (2) as required.

Recall that certain values of Riemann’s \( \zeta \)-function are calculated by Euler in terms of Bernoulli numbers \( B_{2m} \):

\[ 2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m} \]

for an even integer points \( 2m > 2 \).

Theorem 1 thus allows us, in particular, to find the exact numerical values of infinite sums \( Z_f(s) \) for various multiplicative arithmetic functions \( f \).
Exercise 1. Find
\[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \]

Exercise 2. Prove that
\[ \sum_{n=1}^{\infty} \mu(n)^2 n^{-s} = \frac{\zeta(s)}{\zeta(2s)}. \]