Name: __________________________

**Question 1**

a) Let $p$ be an odd prime. Show that the congruence

$$x^2 \equiv 1 \pmod{p^a}$$

has exactly two solutions such that $1 \leq x \leq p^a$ for every integer $a \geq 1$

b) Let $b$ be a positive integer. How many solutions $x$ such that $1 \leq x \leq 2^b$ does the congruence

$$x^2 \equiv 1 \pmod{2^b}$$

have? (Be careful: the answer depends on $b$.)

c) Let $p$ and $q$ be distinct odd primes. How many solutions $x$ such that $1 \leq x \leq pq$ does the congruence

$$x^2 \equiv 1 \pmod{pq}$$

have?
Question 2

Let $p$ be an odd prime.

a) Let $g$ be a primitive root modulo $p$. Prove that

$$g^{\frac{p-1}{2}} \equiv p - 1 \pmod{p}$$

b) Let $G$ be the product of all mutually incongruent primitive roots modulo $p$. Find the residue of $G$ modulo $p$ (that is find $t$ such that $0 \leq t \leq p$ and $G \equiv t \pmod{p}$).
Let $n$ be a positive integer. Let $S$ be the sum of all integers $t$ such that
\[ 1 \leq t \leq n, \quad \text{and} \quad \text{g.c.d.}(t, n) = 1. \]

Find the ratio $S/\phi(n)$. (As usual, we denote by $\phi$ Euler’s $\phi$-function.)
Let $x > 1$ be a positive real number, and $x = [a_0; a_1, a_2, \ldots]$ be its continued fraction representation. Find the continued fraction expansion of the real number $1/x$. 

\textit{Question 4}
Question 5

Solving quadratic equations with continued fractions.

Lagrange’s theorem guarantees that real roots of a quadratic equation have periodic continued fractions representations. We find some of these representations here.

Consider the equation

\[ x^2 - bx - 1 = 0 \]

with a positive integer \( b \). We have

\[ x = b + \frac{1}{x} \]

and we plug in this formula into itself recursively

\[ x = b + \frac{1}{x} = b + \frac{1}{b + \frac{1}{x}} = b + \frac{1}{b + \frac{1}{b + \frac{1}{x}}} = \ldots, \]

obtaining in this way a continued fraction expansion for a positive root of the quadratic equation

\[ x = [b; b, b, b, \ldots]. \]

This way of solving a quadratic equation has, in fact an advantage against the standard one: continued fraction converges faster than alternative algorithms of calculating a square root.

a) Find a continued fraction representation for a root of the quadratic equation

\[ ax^2 - abx - 1 = 0, \]

where \( a \) and \( b \) are positive integers.
b) As we know, for a positive integer $b$, the equation
\[ x^2 - bx - 1 = 0 \]
has two real roots, and $x = [b; b, b, b, \ldots]$ is one of them. Find a continued fraction representation of the second root.

c) Let $n$ be a positive integer. Find a continued fraction representation for the real number (quadratic irrationality)
\[ \sqrt{n^2 + 1}. \]
Question 6

Let

\[ Z(s) = \sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s}, \]

where \( \mu \) is Möbius function. The series converges for \( s > 1 \), and you do not need to prove that. Find \( Z(4) \).

*Hint.* It may be helpful to consider the function \( Z(s)/\zeta(s) \).