Name: _____

Question 1

a) Let p be an odd prime. Show that the congruence

$$x^2 \equiv 1 \pmod{p^a}$$

has exactly two solutions such that $1 \leq x \leq p^a$ for every integer $a \geq 1$

b) Let b be a positive integer. How many solutions x such that $1 \leq x \leq 2^b$ does the congruence

$$x^2 \equiv 1 \pmod{2^b}$$

have? (Be careful: the answer depends on b.)

c) Let p and q be distinct odd primes. How many solutions x such that $1 \leq x \leq pq$ does the congruence

$$x^2 \equiv 1 \pmod{pq}$$

have?

Question 2

Let p be an odd prime.

a) Let g be a primitive root modulo p. Prove that

$$g^{\frac{p-1}{2}} \equiv p-1 \pmod{p}$$

b) Let G be the product of all mutually incongruent primitive roots modulo p. Find the residue of G modulo p (that is find t such that $0 \leq t \leq p$ and $G \equiv t \pmod{p}$).

Question 3

Let n be a positive integer. Let S be the sum of all integers t such that

$$1 \leq t \leq n, \quad \text{and} \quad g.c.d.(t, n) = 1.$$

Find the ratio $S/\phi(n)$. (As usual, we denote by ϕ Euler's ϕ-function.)

Question 4

Let $x > 1$ be a positive real number, and $x = [a_0; a_1, a_2, \ldots]$ be its continued fraction representation. Find the continued fraction expansion of the real number $1/x$.

Question 5

Solving quadratic equations with continued fractions.

Lagrange's theorem guarantees that real roots of a quadratic equation have periodic continued fractions representations. We find some of these representations here.

Consider the equation

$$x^2 - bx - 1 = 0$$

with a positive integer b. We have

$$x = b + \frac{1}{x}$$

and we plug in this formula into itself recursively

$$x = b + \frac{1}{x} = b + \frac{1}{b + \frac{1}{x}} = b + \frac{1}{b + \frac{1}{b + \frac{1}{x}}} = \dots,$$

obtaining in this way a continued fraction expansion for a positive root of the quadratic equation

$$x = [b; b, b, b, \dots].$$

This way of solving a quadratic equation has, in fact an advantage against the standard one: continued fraction converges faster than alternative algorithms of calculating a square root.

a) Find a continued fraction representation for a root of the quadratic equation

$$ax^2 - abx - 1 = 0,$$

where a and b are positive integers.

b) As we know, for a positive integer b, the equation

$$x^2 - bx - 1 = 0$$

has two real roots, and $x = [b; b, b, b, \ldots]$ is one of them. Find a continued fraction representation of the second root.

c) Let n be a positive integer. Find a continued fraction representation for the real number (quadratic irrationality)

$$\sqrt{n^2 + 1}.$$

Question 6

Let

$$Z(s) = \sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s},$$

where μ is Möbius function. The series converges for $s > 1$, and you do not need to prove that. Find $Z(4)$.

Hint. It may be helpful to consider the function $Z(s)/\zeta(s)$.