Name: ________________________________

Question 1

In this question, just write your answers, without proofs. Exercise 14 on p. 342 may be helpful.

a) Let p be a fixed prime integer, and let R be the set of all rational numbers that can be written in a form a/b with b not divisible by p. Note that $R \supset \mathbb{Z}$, because $n = n/1$ for every integer n. One can easily prove (I do not require that) that R is a PID (the addition and multiplication operations are just usual addition and multiplication of rational numbers).

a.1) Describe all units in R.

Answer

a.2) Describe all maximal ideals in R.

Answer

b) Let p be a fixed prime integer, and let R be the set of all rational numbers that can be written in a form a/b with $b = p^k$ for a non-negative integer k. Note that $R \supset \mathbb{Z}$, because $n = n/1$ for every integer n. One can easily prove (I do not require that) that R is a PID (the addition and multiplication operations are just usual addition and multiplication of rational numbers).

b.1) Describe all units in R.

Answer

b.2) Describe all maximal ideals in R.

Answer

Question 2

Let R be a Euclidean domain and let u be a unit in R. We denote by δ the corresponding function. Is it true that $\delta(b) = \delta(ub)$ for every non-zero $b \in R$? Prove your answer.

Answer Yes No (Please circle).

Proof

Question 3

Let $d \in \mathbb{Z}$ be a square-free integer (that is $d \neq 1$, and d has no integer factors of the form c^2 except $c = \pm 1$, cf. p. 346). Let $R = \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$. Our ultimate target in this problem is to prove that every prime ideal $P \subset R$ is a maximal ideal. I provide steps below.

a) We firstly prove that every ideal $I \subset R$ is finitely generated. We actually prove a stronger statement that every ideal is generated by at most two elements. The steps provided below are very similar to those of Exercise 32, p. 344, which may be helpful.

a.1) Prove that if I is non-zero, then $I \cap \mathbb{Z}$ is a non-zero ideal in $\mathbb{Z}$.

a.2) Derive that there exists a positive integer $x \in \mathbb{Z}$ such that

$$I \cap \mathbb{Z} = \{xa \mid a \in \mathbb{Z}\}$$

a.3) Let J be the set of all integers b such that $a + b\sqrt{d} \in I$ for some $a \in \mathbb{Z}$ (that is there exists $a \in \mathbb{Z}$ such that $a + b\sqrt{d} \in I$). Prove that there exists a positive integer y such that

$$J = \{yt \mid t \in \mathbb{Z}\}$$

a.4) Explain why there exists $s \in \mathbb{Z}$ such that $s + y\sqrt{d} \in I$.

a.5) Prove that $I = \{Ax + B(s + y\sqrt{d}) \mid A, B \in \mathbb{Z}\}$. (That implies that $I = (x, s + y\sqrt{d})$ in p.145 notations for finitely generated ideals, but we will not need this fact.)

b) Derive from the statements $a.1 - a.5$ that the factor ring R/P is a finite ring without zero divisors.
b.1) Explain why the factor ring R/P has no zero divisors.

b.2) Prove that the factor ring R/P is finite.

c) Derive from $b.1$ and $b.2$ that R/P is a field.

d) Derive from $a) - d)$ that every prime ideal $P \subset R$ is a maximal ideal.

Question 4

Let $d \geq 6$ be a square free integer (that is $d \neq 1$, and d has no integer factors of the form c^2 except $c = \pm 1$, cf. p. 346). We want to show that the ring

$$R = \mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} \mid a, b \in \mathbb{Z}\}$$

is not a Euclidean domain.

We assume, for the sake of contradiction, that R is a Euclidean domain, and denote by δ the corresponding function (see Definition on p. 324 for its properties).

a) Prove that $\delta(z) > 0$ (that is $\delta(z) \neq 0$) if $z \in R$ is not a unit.

b) Prove that there exists a non-unit element $x \in R$ such that $\delta(x) \leq \delta(z)$ for every non-unit element $z \in R$.

c) Show that a remainder from the division by x is either 0 or a unit in R.

d) Describe all units in R.
Hint. Make use of Theorem 10.20.

e) Derive from d) that for every $w \in R$ at least one of the three elements $w,\ \ w+1$, and $w-1$ must be divisible by x in R.

f) Consider division with remainder of $\sqrt{-d}$ by 2 and make use of c) and d) to show that $x \neq 2$. A very similar argument also shows that $x \neq 3$. You may use this fact further and do not need to write the proof here.

g) Make use of the condition $d \geq 6$ in order to show that one can pick $w \in \mathbb{Z} \subset R$ such that no one of the elements $w,\ \ w+1$, and $w-1$ is divisible by x in R.
Hint. Make use of Theorem 10.19.
The contradiction between the assertions e) and g) finishes the proof.