GENERATING FUNCTIONS: RECURRENCE RELATIONS, RATIONALITY AND HADAMARD PRODUCT.

1. Recurrence relations and rational generating functions

We begin with the following generalization of the Fibonacci sequence.

Definition 1. *A sequence a_n (for $n \geq 0$) is given by a linear recurrence relation of order k with constant coefficients $c_1, \dots, c_k$, if*

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \dots + c_k a_n, \tag{*}$$

and the numbers $a_0, \dots, a_{k-1}$ are given.

Using this terminology, we say that the sequence of Fibonacci numbers is given by a linear recurrence relation of order 2. Recall that a rational function is a quotient of two polynomial functions. In particular, the generation function for Fibonacci numbers is rational. This fact may be generalized as follows.

Theorem 1. *Suppose a sequence is given by a linear recurrence relation* (*)*. Then the generating function*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

is rational,

$$A(x) = \frac{P(x)}{Q(x)},$$

where P is a polynomial of degree at most $k-1$, and

$$Q(x) = 1 - \sum_{n=1}^{k} c_k x^k = 1 - c_1 x - c_2 x^2 - \dots - c_k x^k$$

Remark 1. Amusingly, $1 - Q(x)$ may be viewed as the generating function for the (finitely many) numbers c_n which determine the linear recurrence relation (*).

The proof of both this and next Theorems are based on the following technical Lemma which helps us to do certain manipulations with power series.

Lemma 1. *For a polynomial $c_1 x + \dots + c_k x^k$ and a power series $\sum_{n=0}^{\infty} a_n x^n$ we have*

$$(c_1 x + \dots + c_k x^k) \sum_{n=0}^{\infty} a_n x^n = R + \sum_{n=0}^{\infty} (c_1 a_{n+k-1} + \dots + c_k a_m)\, x^{n+k}$$

with a polynomial R of degree at most $k-1$.

Remark 2. There is nothing mysterious about R, and it is not difficult to calculate this polynomial. Moreover, it is not difficult to carry over this calculation. We do not do that here only because we do not need that for our proofs. However, explicit knowledge of this polynomial may be quite useful in specific cases.

Proof. The statement follows immediately from the following calculation.

$$
\begin{aligned}
&(c_1 + c_2x^2 + \ldots + c_kx^k) \sum_{n=0}^{\infty} a_nx^n \\
&\qquad = \quad c_1x \left(\sum_{n=0}^{k-2} a_nx^n \quad + \sum_{n=k-1}^{\infty} a_nx^n \right) \\
&\qquad + \quad c_2x^2 \left(\sum_{n=0}^{k-3} a_nx^n \quad + \sum_{n=k-2}^{\infty} a_nx^n \right) \\
&\qquad \qquad \cdots \\
&\qquad \qquad \cdots \\
&\qquad \qquad \cdots \\
&\qquad + \quad c_{k-1}x^{k-1} \left(\sum_{n=0}^{0} a_nx^n \quad + \sum_{n=1}^{\infty} a_nx^n \right) \\
&\qquad + \quad c_kx^k \left(\sum_{n=0}^{-1} a_nx^n \quad + \sum_{n=0}^{\infty} a_nx^n \right),
\end{aligned}
$$

where we assumed that $\sum_{n=0}^{0} a_nx^n = a_0$, and $\sum_{n=0}^{-1} a_nx^n = 0$. We now execute the multiplication by c_jx^j in every row above, and take into the account that

$$c_jx^j \sum_{n=k-j}^{\infty} a_nx^n = \sum_{n=k}^{\infty} c_ja_{n-j}x^n.$$

We obtain the identity claimed in the lemma with

$$R = c_1 \sum_{n=0}^{k-2} a_nx^{n+1} + c_2 \sum_{n=0}^{k-3} a_nx^{n+2} + \ldots + c_{k-1}a_0x^{k-1},$$

a polynomial of degree at most $k - 1$.

□

Proof of Theorem 1. Using Lemma 1 and recursion (*) we obtain that

$$
\begin{aligned}
(1 - Q) \sum_{n=0}^{\infty} a_nx^n &= R + \sum_{n=0}^{\infty} (c_1a_{n+k-1} + \ldots + c_ka_m)\, x^{n+k} \\
&= R + \sum_{n=0}^{\infty} a_{n+k}x^{n+k} = R + \sum_{n=k}^{\infty} a_nx^n \\
&= R - \sum_{n=0}^{k-1} a_nx^n + \sum_{n=0}^{k-1} a_nx^n + \sum_{n=k}^{\infty} a_nx^n \\
&= -P + \sum_{n=0}^{\infty} a_nx^n,
\end{aligned}
$$

where $P = -R + \sum_{n=0}^{k-1} a_nx^n$ is a polynomial of degree at most $k - 1$. Rewriting last equation as

$$(1 - Q) \sum_{n=0}^{\infty} a_nx^n = -P + \sum_{n=0}^{\infty} a_nx^n,$$

we deduce the statement of Theorem 1 □

The converse theorem is also true, and we will prove it. However, we need to make a few remarks in order to correctly and reasonably state the converse theorem. Roughly speaking, we want to prove that every rational function is a generating function for a sequence of numbers which satisfy a linear recurrence relation. Let

$$B(x) = \frac{U(x)}{W(x)}$$

with polynomials U and W be a rational function. It has a Laurent expansion in x

$$B(x) = \sum_{n=M}^{\infty} b_n x^n,$$

thus $B(x)$ is a generating function for the sequence of numbers b_n. It is reasonable to and we will assume that polynomials U and W are muturally prime (that is they do not have common complex roots), because otherwise we can cancel their common factors without altering the sequence $\{b_n\}$. If $W(0) = 0$, then $M < 0$, and $W(x) = x^{-M}V(x)$ with a polynomial $V(x)$ such that $V(0) \neq 0$. Thus

$$x^{-M}B(x) = \frac{U(x)}{V(x)} = \sum_{n=M}^{\infty} b_n x^{n-M} = \sum_{n=0}^{\infty} b_{n+M} x^n,$$

and we may study the properties of the sequence $d_n = b_{n+M}$ instead of b_n since these two sequences consists of the same quantities in the same order. If the $degU \geq degV$, we can perform the long division

$$\frac{U(x)}{V(x)} = S(x) + \frac{P(x)}{V(x)}$$

with a polynomial $S(x)$ and polynomial P such that $degP < degV$. The rational function

$$\frac{P(x)}{V(x)} = x^{-M}B(x) - S(x)$$

is a generating function for essentially the same sequence d_n; only finitely many first quantities are altered by subtraction of polynomial S. Finally, we let

$$Q(x) = \frac{V(x)}{V(0)}$$

and

$$V(0)(x^{-M}B(x) - S(x)) = \frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} a_n x^n. \tag{1}$$

Exercise 1. *Assume that sequence $\{a_b\}$ is given by a linear recurrence relation. What can be derived about sequence $\{b_n\}$ from this fact?*

Note that $Q(x)$ is a polynomial, and $Q(0) - 1$. We now introduce a notation for its coefficients

$$Q(x) = 1 - \sum_{n=1}^{k} c_k x^k = 1 - c_1 x - c_2 x^2 - \ldots - c_k x^k.$$

Theorem 2. *Assume the notations and conventions above. Then the sequence $\{a_n\}$ satisfies a linear recurrence relation*

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \ldots + c_k a_n \tag{*}$$

for $n \geq 0$.

Proof. Definition of the sequence $\{a_n\}$ implies that

$$Q(x) \sum_{n=0}^{\infty} a_n x^n = P(x),$$

and, therefore,

$$(1 - Q(x)) \sum_{n=0}^{\infty} a_n x^n = -P(x) + \sum_{n=0}^{\infty} a_n x^n$$

Take into the account that $1 - Q(x) = c_1 x + \ldots + c_k x^k$, and apply Lemma 1 to the left-hand side of this identity to obtain

$$R(x) + \sum_{n=0}^{\infty} (c_1 a_{n+k-1} + \ldots + c_k a_m)\, x^{n+k} = -P(x) + \sum_{n=0}^{\infty} a_n x^n$$

with a polynomial $R(x)$ such that $degR \leq k-1$. Note that $degP < degV = degQ = k$. It follows that the difference between the two infinite series

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} (c_1 a_{n+k-1} + \ldots + c_k a_m)\, x^{n+k} = R(x) + P(x)$$

is a polynomial of degree at most $k - 1$. Thus all coefficients of x^n for $n \geq k$ in this difference are zeros, and implies the statement of Theorem 2 □

2. Hadamard product of rational generating functions

We begin with a technical lemma which provides us another characterization of sequences whose generating functions are rational.

Lemma 2. *Let* $\{a_n\}$ *be a sequence. The generating function*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a rational function if and only if there exist an integer l*, a set of* l *numbers* $q_1, \ldots q_l$*, and a set of* l *polynomials* $p_1, \ldots, p_l$ *such that*

$$a_n = p_1(n) q_1^n + \ldots + p_l(n) q_l^n \tag{2}$$

for all n *big enough.*

Remark 3. The words "for all n big enough" mean that there exist M such that the statement is true for all $n > M$.

Proof. We firstly assume that the function $A(x)$ is rational.

Exercise 2. *Explain why it suffices to consider the case when* $A(x) = P(x)/Q(x)$ *with polynomials* $P(x)$ *and* $Q(x)$ *such that* $degP < degQ$ *and* $Q(0) \neq 0$.

We apply partial fraction decomposition to represent $A(x)$ as a linear combination

$$A(x) = \sum_{j=1}^{N} \frac{s_j}{(1 - q_j x)^{k_j}}, \tag{3}$$

where $1/q_j$ are roots of the polynomial $Q(x)$.

Remark 4. Both roots of $Q(x)$ and coefficients s_j may be complex numbers, while the degrees k_j are positive integers. We do not prove here the existence of the decomposition (3): this is an easy fact from complex analysis.

Exercise 3. *Explain why, in view of decomposition (3), it suffices to prove the claim of Lemma 2 for a rational function*

$$A(x) = \frac{1}{(1-qx)^k}. \tag{4}$$

We now assume (4) with a positive integer k and a complex number q.

Exercise 4. *Prove that*

$$\frac{1}{(1-qx)^k} = \sum_{n=0}^{\infty} \binom{k+n-1}{k-1} q^n x^n.$$

Exercise 5. *Use the well-known formula*

$$\binom{k+n-1}{k-1} = \frac{1}{(k-1)!}\ (n+1)(n+2)\dots(n+k-1)$$

to finish the proof of (2).

We now turn to the proof of the converse statement. Namely, we assume that (2) holds for all $n > M$, and prove that

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a rational function.

Exercise 6. *Show that there is a polynomial $S(x)$ of degree at most M such that for the function*

$$B(x) = A(x) - S(x) = \sum_{n=0}^{\infty} b_n x^n$$

we have

$$b_n = p_1(n)q_1^n + \dots + p_l(n)q_l^n$$

for all $n \geq 0$, and $a_n = b_n$ for all $n > M$.

If $B(x)$ is a rational function, then so is $A(x) = B(x) + S(x)$. It therefore suffices to prove that $B(x)$ is a rational function.

Exercise 7. *Show that it is sufficient to consider a special case*

$$b_n = p(n)q^n \tag{5}$$

for a polynomial $p(x)$.

We now assume (5), and let $k = deg\ p(x)$. Let $P_1 = 1/(k-1)!$, and for an integer $j \geq 2$, let

$$P_j(x) = \frac{1}{(k-1)!}(x+1)(x+2)\dots(x+j-1).$$

Since P_j is a polynomial of degree $deg P_j = j - 1$, the polynomials $P_1, \dots, P_{k+1}$ for a basis of the linear space of all polynomials of degree at most k. (This is an easy fact from linear algebra.) Thus there exist complex numbers $\beta_1, \dots \beta_{k+1}$ such that

$$p(x) = \beta_1 P_1(x) + \dots + \beta_{k+1} P_{k+1}(x).$$

Exercise 8. *Use the result from exercise 4 and the formula from 5 to prove that*

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = \beta_1 \frac{1}{(1-qx)} + \ldots + \beta_{k+1} \frac{1}{(1-qx)^{k+1}}.$$

We conclude that $B(x)$ is a rational function, and the lemma is now proved. □

We now define Hadamard product of sequences.

Definition 2. *Let $\{a_n\}$ and $\{b_n\}$ (for $n \geq 0$) be two sequences. Hadamard product of $\{a_n\}$ and $\{b_n\}$ is the sequence of products $\{a_n b_n\}$.*

The following amazing result makes the class of sequences with rational generating functions especially interesting.

Theorem 3. *Assume that the generating functions for the sequences $\{a_n\}$ and $\{b_n\}$*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad B(x) = \sum_{n=0}^{\infty} b_n x^n$$

are rational. Then the generating function for their Hadamard product

$$C(x) = \sum_{n=0}^{\infty} a_n b_n x^n$$

is also rational.

Proof. It follows from Lemma 1 that there exist two sets of polynomials $p_1, \ldots, p_l$ and $r_1, \ldots r_m$, two sets of numbers $q_1, \ldots, q_m$ and $s_1, \ldots, s_m$ such that, for n big enough,

$$a_n = p_1(n) q_1^n + \ldots + p_l(n) q_l^n$$

and

$$b_n = r_1(n) s_1^n + \ldots + r_m(n) s_m^n$$

Define polynomials $P_{i,j}(x) = p_i(x) r_j(x)$, and numbers $q_{i,j} = q_i s_j$. It follows that

$$a_n b_n = \sum_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} P_{i,j}(n) q_{i,j}^n.$$

for n big enough. The rationality of the generating function $C(x)$ follows from this expression and Lemma 1 □