

SOME CONGRUENCES FOR TRACES OF SINGULAR MODULI

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ABSTRACT. We address a question posed by Ono [7, Problem 7.30], prove a general result for powers of an arbitrary prime, and provide an explanation for the appearance of higher congruence moduli for certain small primes. One of our results overlaps but does not coincide with a recent result of Jenkins [6]. This result essentially coincides with a recent result of Edixhoven [3], and we hope that the comparison of the methods, which are entirely different, may reveal a connection between the p -adic geometry and the arithmetic of half-integral weight Hecke operators.

1. INTRODUCTION AND DISCUSSION OF THE RESULTS

Throughout the paper D and d denote positive and non-negative integers, respectively, which satisfy the congruences

$$D \equiv 0, 1 \pmod{4} \quad \text{and} \quad d \equiv 0, 3 \pmod{4}.$$

We denote by $\chi_d = \left(\frac{-d}{\cdot}\right)$ and $\chi_D = \left(\frac{D}{\cdot}\right)$ the quadratic Dirichlet characters associated with the imaginary and real quadratic fields $\mathbb{Q}(\sqrt{-d})$ and $\mathbb{Q}(\sqrt{D})$ correspondingly. The character χ_d (resp. χ_D) is primitive if $-d$ (resp. D) is a fundamental discriminant.

Zagier considered in [9] the nearly holomorphic modular forms on $\Gamma_0(4)$

$$f_d = q^{-d} + \sum_{D>0} A(D, d)q^D$$

of weight $1/2$ and

$$g_D = q^{-D} + \sum_{d \geq 0} B(D, d)q^d.$$

of weight $3/2$. (Here and in the following $q = \exp(2\pi i\tau)$ with $\Im(\tau) > 0$.) The absence of holomorphic modular forms on $\Gamma_0(4)$ of weights $3/2$ and $1/2$ which satisfy Kohnen's plus-condition and have zero constant term

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implies the uniqueness of f_d and g_D . The explicit recursive construction of f_d and g_D , provided by Zagier, guarantees that $A(D, d), B(D, d) \in \mathbb{Z}$.

For an integer $m \geq 1$ the Hecke operators $T(m)$ act on these forms and preserve the integrality of the Fourier coefficients. Following [9] we denote by $A_m(D, d)$ and $B_m(D, d)$ the coefficient of q^D in $f_d|_{\frac{1}{2}}T(m)$ and the coefficient of q^d in $g_D|_{\frac{3}{2}}T(m)$, respectively. Zagier proved that

$$(1) \quad A_m(D, d) = -B_m(D, d),$$

and this common value divided by \sqrt{D} is the (twisted if $D > 1$) trace of a certain modular function. This interpretation in terms of traces of singular moduli is a primary source of motivation for the investigation of these numbers.

Ahlgren and Ono studied the arithmetic of traces of singular moduli in [1], and, in particular, proved the congruences

$$(2) \quad A_m(1, p^2d) \equiv 0 \pmod{p}$$

if the prime p splits in $\mathbb{Q}(\sqrt{-d})$ and $p \nmid m$. In this connection Ono posed a question [7, Problem 7.30] whether there are natural generalizations of (2) modulo arbitrary powers of p . Numerical evidence indicates that if $\chi_d(p) = \chi_D(p)$, then

$$(3) \quad A_m(D, p^{2n}d) \equiv 0 \pmod{p^n}$$

with maximum congruence modulus exceeding p^n for $p \leq 11$.

The question splits into two parts: to find similar congruences which hold for powers of an arbitrary prime and to find series of stronger congruences for special primes.

We firstly comment on the former part of the question. Recently Edixhoven [3] used the interpretation of the numbers $A_m(1, p^{2n}d)$ as traces of singular moduli and the local moduli theory of ordinary elliptic curves in positive characteristic and obtained the following result.

Theorem (Edixhoven). *If $D = 1$ and $\chi_d(p) = 1$, then (3) holds for any $m \geq 1$.*

Recently Jenkins [6] presented an elementary argument based on the identity (1) and standard formulas for the action of half-integral weight Hecke operators. Jenkins' result recovers the congruences obtained by Edixhoven in the case $m = 1$. More precisely, he proves the following.

Theorem (Jenkins). *If $\chi_d(p) = \chi_D(p) \neq 0$, then*

$$(4) \quad A(D, p^{2n}d) = p^n A(p^{2n}D, d),$$

and, therefore, (3) holds for $m = 1$.

In this paper we prove the following congruences.

Theorem 1. *Let $-d$ and D be fundamental discriminants.*

- a. *If $\chi_d(p) = \chi_D(p)$, then (3) holds for any $m \geq 1$.*
- b. *If $\chi_d(p) = -\chi_D(p) \neq 0$, then for any $m \geq 1$*

$$A_m(D, p^{2n+2}d) - A_m(D, p^{2n}d) \equiv 0 \pmod{p^n}.$$

Our argument is also elementary and uses nothing but the identity (1) and some facts about the action of half-integral weight Hecke operators. The assumption that both discriminants are fundamental allows us to present a simple general argument. For instance, Jenkins' identity (4) under this assumption follows at once from the definitions (see (9) below). It is, however, likely that (3) is true without this assumption. Note that our Theorem 1a essentially coincides with the result of Edixhoven. This provides a reason to speculate that the arithmetic of half-integral weight Hecke operators is somehow connected with the p -adic geometry. Such a connection, if it really exists, looks like an enticing subject to investigate.

We now turn to the latter part of Ono's question. He noticed [7, Example 7.15] that, for $p \leq 11$, the maximum congruence modulus in (2) exceeds p and called for explanations. Recently Boylan [2] found a fairly exact answer in the case $p = 2$. Combining the result of Jenkins, a recent result of the author [5] and a theorem of Serre we prove the following qualitative result, which uniformly explains the phenomenon without providing information about the specific congruence moduli.

Theorem 2. *Let $p \leq 11$. If $\chi_d(p) = \chi_D(p) \neq 0$, then the p -adic limit*

$$\lim_{n \rightarrow \infty} p^{-n} A(D, p^{2n}d) = 0.$$

2. PROOFS

Theorem 1 follows at once from the following proposition.

Proposition 1. *Let p be a prime, and let $m \geq 1$ and $n \geq 0$ be integers. Let $-d$ and D be fundamental discriminants (i.e. $-d$ is the discriminant of $\mathbb{Q}(\sqrt{-d})$, and D is either 1 or the discriminant of $\mathbb{Q}(\sqrt{D})$).*

- a. *If $\chi_d(p) = \chi_D(p)$ then*

$$(5) \quad A_m(D, p^{2n}d) = p^n A_m(p^{2n}D, d).$$

- b. *If $\chi_d(p) = -\chi_D(p) \neq 0$ then*

$$A_m(D, p^{2n+2}d) - A_m(D, p^{2n}d) = p^{n+1} A_m(p^{2n+2}D, d) + p^n A_m(p^{2n}D, d).$$

Proof of Proposition 1. Let $F = \sum b(n)q^n$ be a (holomorphic or nearly holomorphic) modular form of weight $k + 1/2$ for an integer $k \geq 0$, which belongs to the Kohnen plus-space (i.e. $b(n) = 0$ if $(-1)^k n \equiv 2, 3 \pmod{4}$). The Hecke operator $T^+(m)$ acts on the Kohnen plus-space and sends F to $F|_{k+1/2}T^+(m) = \sum b_m^+(n)q^n$. If $(-1)^k n$ is a fundamental discriminant, then

$$(6) \quad b_m^+(n) = \sum_{l|m} \left(\frac{(-1)^k n}{l} \right) l^{k-1} b \left(\frac{m^2}{l^2} n \right).$$

This formula follows from [4, Th. 4.5], where it is proved in the equivalent language of Jacobi forms. Although formally the quoted theorem applies only to holomorphic half-integral weight modular forms, as it is mentioned in [9], nothing changes if we allow the pole at infinity. Also note that the technique of Jacobi forms from [4] covers only the case of odd k . However, if $k = 0$ (the only even k which we need here) an equivalent formula may be found in Zagier's paper [9, proof of Th. 7]. If $k = 0$, non-trivial denominators apparently appear in the right-hand side of (6). In order to get rid of these denominators and to keep our notations compatible with those of [9] we renormalize the Hecke operators by

$$(7) \quad T(m) = \begin{cases} mT^+(m) & \text{if } k = 0 \\ T^+(m) & \text{otherwise.} \end{cases}$$

Zagier used the operators $T(m) = mT^+(m)$ in the definition of the quantities $A_m(D, d)$ (cf. [9, formula (17)]). It follows from (6) and the definition of the quantities $A_{p^n}(D, d)$ and $B_{p^n}(D, d)$ that under the assumptions of Proposition 1

$$(8) \quad A_{p^n}(D, d) = \sum_{i=0}^n \chi_D(p^{n-i}) p^i A(p^{2i}D, d)$$

$$B_{p^n}(D, d) = \sum_{i=0}^n \chi_d(p^{n-i}) B(D, p^{2i}d).$$

These equations combined with (1) imply that for any $n \geq 0$

$$(9) \quad \sum_{i=0}^n \chi_D(p^{n-i}) p^i A(p^{2i}D, d) = \sum_{i=0}^n \chi_d(p^{n-i}) A(D, p^{2i}d),$$

and an induction argument in n finishes the proof of Proposition 1 in the case when $m = 1$.

We now generalize the above argument to the case of arbitrary integer $m \geq 1$. Recall the usual relation between Hecke operators acting on the Kohnen plus-space of modular forms of weight $k + 1/2$ (see i.e. [4, Cor. 1 to Th. 4.5]):

$$(10) \quad T^+(u)T^+(u') = \sum_{c|(u,u')} c^{2k-1}T^+(uu'/c^2).$$

In particular, for integers $n, s \geq 0$ we have in view of the definition (7) uniformly for $k = 0, 1$

$$\sum_{i=0}^{\min(n,s)} p^i T(p^{n+s-2i}) = T(p^n)T(p^s),$$

and the equations (8) generalize to

$$(11) \quad \sum_{i=0}^{\min(n,s)} p^i A_{p^{n+s-2i}}(D, d) = \sum_{i=0}^n \chi_D(p^{n-i}) p^i A_{p^s}(p^{2i}D, d)$$

$$\sum_{i=0}^{\min(n,s)} p^i B_{p^{n+s-2i}}(D, d) = \sum_{i=0}^n \chi_d(p^{n-i}) B_{p^s}(D, p^{2i}d).$$

As previously, (1) implies that the left-hand sides of (11) are equal in absolute value and have opposite signs, and we obtain the following generalization of (9) for $n, s \geq 0$

$$(12) \quad \sum_{i=0}^n \chi_D(p^{n-i}) p^i A_{p^s}(p^{2i}D, d) = \sum_{i=0}^n \chi_d(p^{n-i}) A_{p^s}(D, p^{2i}d).$$

An induction argument in n finishes the proof of Proposition 1 in the case when $m = p^s$ with $s > 0$.

Assume now that $m = p^s m_0$ with $p \nmid m_0$ and $s \geq 0$. It follows from (10) and (7) that

$$T(p^s m_0) = T(p^s)T(m_0)$$

for any $k \geq 0$. It follows that we can multiply the indices in (12) and (11) by m_0 , as we have had begun the whole argument with the consideration of $f_d|_{\frac{1}{2}}T(m_0)$ and $g_D|_{\frac{3}{2}}T(m_0)$ instead of f_d and g_D correspondingly. This implies the following generalization of (9),(12) for $m_0 \geq 1$, $p \nmid m_0$, $s \geq 0$ and any $n \geq 0$:

$$\sum_{i=0}^n \chi_D(p^{n-i}) p^i A_{m_0 p^s}(p^{2i}D, d) = \sum_{i=0}^n \chi_d(p^{n-i}) A_{m_0 p^s}(D, p^{2i}d),$$

and an induction argument in n completes the proof of Proposition 1.

□

Proof of Theorem 2. It follows from [5, Theorem 2.8b] that, since $p \nmid dD$, the formal power series in q

$$F = \sum_{\substack{n>0 \\ p|n}} \sum_{\substack{l|n \\ (p,l)=1}} \left(\frac{D}{l}\right) l^{-1} A\left(\frac{n^2}{l^2}D, d\right) q^n$$

is a p -adic cusp form of weight 0. A result of Serre [8, Th. 7; Rem., p.216] implies that, for $p \leq 11$, the p -adic limit

$$\lim_{n \rightarrow \infty} F|U^n = 0,$$

where U denotes Atkin's U -operator ($\sum a(n)q^n|U = \sum a(pn)q^n$). Thus the coefficient of q^{p^n} in F approaches 0 p -adically as $n \rightarrow \infty$. That is

$$\lim_{n \rightarrow \infty} A(p^{2n}D, d) = 0.$$

The latter equality combined with Jenkins' identity (4) completes the proof of Theorem 2. □

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