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ON THE HECKE EQUIVARIANCE OF THE BORCHERDS ISOMORPHISM

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Abstract

We prove that the Borcherds isomorphism is Hecke equivariant if one considers multiplicative Hecke operators acting on the integral weight meromorphic modular forms. This answers a part of a question of Borcherds [**1**, 17.10] using his suggestion to define the multiplicative Hecke operators.

Let $\mathcal{M}$ be the multiplicative group of integral weight meromorphic modular forms for some character of $SL_2(\mathbb{Z})$ with integer coefficients, leading coefficient 1, and whose zeros and poles are supported at cusps and imaginary quadratic irrationals. Denote by $\mathcal{M}_{k,h} \subset \mathcal{M}$ the subset which consists of modular forms of weight k and for which order of the pole at infinity is h. We have $\mathcal{M} = \cup_{k\in\mathbb{Z}, h\in\frac{1}{12}\mathbb{Z}} \mathcal{M}_{k,h}$ and $\mathcal{M}_{k_1,h_1}\mathcal{M}_{k_2,h_2} \subset \mathcal{M}_{k_1+k_2,h_1+h_2}$. Let $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ be the space of modular forms of weight $1/2$ with respect to $\Gamma_0(4)$ which satisfy Kohnen's plus-condition and whose poles are supported at the cusps of $\Gamma_0(4)$. (Recall that a modular form $f(\tau)$ of weight $1/2$ satisfies Kohnen's plus-condition if its q-expansion $\sum_{n\gg-\infty} c(n)q^n$ has $c(n) = 0$ for $n \equiv 2, 3 \bmod 4$. Throughout, we let $q = e^{2\pi i\tau}$ with $\Im\tau > 0$.) In [**1**, Theorem 14.1], Borcherds establishes an isomorphism between the multiplicative group $\mathcal{M}$ and the additive group $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$. Denote this isomorphism by $B\colon \mathcal{M} \to \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$.

Let p be a prime. Let k be an integer and let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real 2×2 matrix whose determinant is positive. The operator $[\alpha]_k$ acts on meromorphic functions on the complex upper half-plane by

$$f \circ [\alpha]_k(\tau) = f((a\tau+b)/(c\tau+d))(c\tau+d)^{-k}(\det\alpha)^{k/2}.$$

Let A^p be the set of 2×2 matrices with integer entries and whose determinant equals p. Consider the coset decomposition

$$A^p = \sum_{i=1}^{p+1} SL_2(\mathbb{Z})\alpha_i. \tag{0.1}$$

Define the multiplicative Hecke operator $M(p)$ by

$$M_{k,h}(p)f = M(p)f = \varepsilon p^{k(p-1)/2} \prod_{i=1}^{p+1} f \circ [\alpha_i]_k, \tag{0.2}$$

where $\varepsilon = e^{\pi i h(p-1)}$ is a twelfth root of unity chosen such that the leading coefficient of $M(p)f$ is 1. It is straightforward to check that $M(p)f$ does not depend on the choice of representatives $[\alpha_i]$. Also $M(p)$ maps elements of $\mathcal{M}_{k,h}$ to

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elements of $\mathcal{M}_{(p+1)k,(p+1)h}$ and defines a doubly graded multiplicative group endomorphism: if $f, g \in \mathcal{M}$ have weights k_1 and k_2 have poles at infinity of orders h_1 and h_2 respectively, then $M_{k_1+k_2,h_1+h_2}(p)(fg) = M_{k_1,h_1}(p)fM_{k_2,h_2}(p)g \in \mathcal{M}_{(p+1)(k_1+k_2),(p+1)(h_1+h_2)}$.

On the weight $\frac{1}{2}$ side let $T^+_{1/2}(p^2)$ be the usual Hecke operator. If $\phi \in \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ has the Fourier expansion $\phi = \sum_{n\gg-\infty} c(n)q^n$ then $T^+_{1/2}(p^2)\phi = \sum_{n\gg-\infty} b(n)q^n$ with

$$b(n) = c(p^2n) + p^{-1}\left(\frac{n}{p}\right)c(n) + p^{-1}c(n/p^2). \tag{0.3}$$

This is a direct extension to the meromorphic case of the definition of Hecke operators acting on the Kohnen's plus-subspace of holomorphic half-integral weight modular forms [**3**]. The proof that $pT^+_{1/2}(p^2)$ acts on $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ repeats the corresponding proof in the holomorphic case [**3**, Theorem 1 i]. Note that $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ is a graded additive group, the gradation being defined by $c(0)$, the constant term, and $pT^+_{1/2}(p^2)$ is a graded group endomorphism. Similarly to the multiplicative case above the endomorphism multiplies the gradation by $p+1$. Under the Borcherds isomorphism $c(0)$ corresponds to the weight of a modular form.

We prove that the Borcherds group isomorphism B commutes with the action of Hecke operators.

Theorem 1. *The diagram of graded groups and their homomorphisms*

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "B"] \arrow[d, "M(p)"'] & \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4)) \arrow[d, "pT^+_{1/2}(p^2)"] \\
\mathcal{M} \arrow[r, "B"] & \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))
\end{tikzcd}$$

is commutative.

Proof. Let $f \in \mathcal{M}_k$ and consider its Borcherds product expansion

$$f = q^{-h}\prod_{n\geq 1}(1-q^n)^{c(n^2)}.$$

According to [**1**, Theorem 14.1], there is a unique $\phi = Bf \in \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ with the Fourier expansion $Bf = \sum_{m\gg-\infty} c(n)q^n$. Put $pT^+_{1/2}(p^2)\phi = \sum_{n\gg-\infty} pb(n)q^n$ with the coefficients $b(n)$ defined by (0.3). Pick the matrices $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$ with $j = 0, \ldots, p-1$ as the representatives in (0.1). Put $\zeta = e^{2\pi i/p}$. We have

$$\begin{aligned}
M(p)f &= \varepsilon f \circ \left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right]_k \prod_{j=0}^{p-1} f \circ \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \\
&= \varepsilon q^{-hp} \prod_{j=0}^{p-1} \zeta^{-hj} q^{-h/p} \prod_{n\geq 1} \left((1-q^{np}) \prod_{j=0}^{p-1} (1-\zeta^{nj} q^{n/p}) \right)^{c(n^2)} \\
&= q^{-h(p+1)} \prod_{\substack{n\geq 1 \\ (p,n)=1}} (1-q^n)^{c(n^2)} (1-q^{np})^{c(n^2)} \times \\
&\qquad\qquad \prod_{n\geq 1} (1-q^n)^{pc(p^2n^2)} (1-q^{p^2n})^{c(p^2n^2)} \\
&= q^{-h(p+1)} \prod_{n\geq 1} (1-q^n)^{pb(n^2)},
\end{aligned}$$

where the one before last equality follows from the identity $\prod_{j=0}^{p-1}(1-\zeta^j X) = 1 - X^p$. Again by Borcherds theorem [**1**, Theorem 14.1] the element of $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ which corresponds to $M(p)f \in \mathcal{M}$ is uniquely defined by the product expansion exponents $pb(n^2)$, and therefore coincides with $pT^+_{1/2}(p^2)\phi$. $\square$

Remark 1. The calculation above also proves that our multiplicative Hecke operator $M(p)$ takes modular forms with integer coefficients and leading coefficient equal to one to modular forms which have the same properties. In particular, this explains our choice of the factor $p^{k(p-1)/2}$ in the definition (0.2) instead of the usual factor $p^{k/2-1}$ for the classical additive Hecke operators.

Remark 2. Since the isomorphism B was discovered, lots of research of various mathematicians was devoted to generalizations of the Borcherds construction. The author admits the possibility that this result is known to the experts. For instance, as the referee pointed out, Gritsenko and Nikulin proved in [**2**, Theorem A.7] an analogous result for the Borcherds lift for $Sp(2,\mathbb{Z})$. However, our result never appeared in the literature. The author chooses to present the result to the readers because the proof is simple, fits well into the original setting of Borcherds [**1**, Theorem 14.1] and answers a question posed there [**1**, 17.10] exactly as it was formulated.

Remark 3. One may inquire about a more intrinsic description of ε. The modular form f belongs to a character of $SL_2(\mathbb{Z})$. Such a character is defined by its values on elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ which must be a $e^{\pi i a}$ and $e^{2\pi i b/3}$ correspondingly with $a = 0, 1$ and $b = 0, 1, 2$. From Zagier's explicit construction of the Borcherds isomorphism [**4**] one derives that the denominator of h coincides with the denominator of $a/2 + b/3$. It follows that $\varepsilon = e^{\pi i (p-1)(3a+2b)/6}$.

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