

The Borcherds-Zagier Isomorphism and a p -Adic Version of the Kohnen-Shimura Map

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1 Introduction

Let \mathcal{M} be the space of even integer weight meromorphic modular forms on $SL_2(\mathbb{Z})$ with integer coefficients, leading coefficient equal to one, and whose zeros and poles are supported at cusps and imaginary quadratic irrationals. If $r \geq 0$ is an integer, let $\mathcal{M}_{r+1/2}^+(\Gamma_0(4))$ be the space of modular forms of half-integral weight $r + 1/2$ with respect to $\Gamma_0(4)$ which satisfy Kohnen's plus-condition and whose poles are supported at the cusps of $\Gamma_0(4)$. (Recall that a modular form $f(\tau)$ satisfies Kohnen's plus-condition if its q -expansion $\sum_{n \geq 0} c(n)q^n$ has $c(n) = 0$ if $(-1)^r n \equiv 2, 3 \pmod{4}$.) In [3, Theorem 14.1], Borcherds establishes an isomorphism between the multiplicative group \mathcal{M} and the additive group $\mathcal{M}_{1/2}^+(\Gamma_0(4))$. For example, an explicit construction of the Borcherds isomorphism is obtained by Zagier in [16], as we now describe. Using Zagier's notation, if $d \equiv 0, 3 \pmod{4}$ is a positive integer, we denote by \mathcal{Q}_d the set of positive definite binary quadratic forms $Q = [a, b, c] = aU^2 + bUV + cV^2$ ($a, b, c \in \mathbb{Z}$) with discriminant $b^2 - 4ac = -d$ with the usual action of the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$ on \mathcal{Q}_d . To each $Q \in \mathcal{Q}_d$ we associate its unique root α_Q in the upper half-plane and put $w_Q = 2$ or 3 if Q is Γ -equivalent to $[a, 0, a]$ or $[a, a, a]$, and 1 otherwise. Let $j = q^{-1} + 744 + 196884q + \dots$ denote the modular invariant ($q = \exp(2\pi i\tau)$ throughout). Consider the weight 0 modular form

$$\mathcal{H}_d = \prod_{Q \in \mathcal{Q}_d/\Gamma} (j(\tau) - j(\alpha_Q))^{1/w_Q} = q^{-H(d)} \prod_{n \geq 1} (1 - q^n)^{\Lambda(n^2, d)}, \quad (1.1)$$

where $H(d) = \sum_{Q \in \Omega_d/\Gamma} w_Q^{-1}$ is the Hurwitz-Kronecker class number. The rational integers $A(n^2, d)$ are defined by the Borcherds product on the right-hand side. In [16], Zagier shows that

$$f_d = q^{-d} + \sum_{D>0} A(D, d)q^D \tag{1.2}$$

is the unique element in $\mathcal{M}_{1/2}^+(\Gamma_0(4))$ with q -expansion $f_d(\tau) = q^{-d} + \mathcal{O}(q)$, and that f_d is connected to $\mathcal{H}_d \in \mathcal{M}$ via the Borcherds isomorphism.

The Kohnen-Shimura map

$$\mathcal{L}_{\delta,r} : \sum_{n \geq 0} b(n)q^n \mapsto \frac{b(0)}{2} L\left(1-r, \left(\frac{\delta}{\cdot}\right)\right) + \sum_{n>0} \sum_{m|n} \left(\frac{\delta}{m}\right) m^{r-1} b\left(\frac{n^2}{m^2}|\delta|\right) q^n \tag{1.3}$$

was discovered by Kohnen [10] for $(\delta, r) \neq (1, 0)$ and a fundamental discriminant δ with $(-1)^r \delta > 0$ as a refinement of Shimura correspondence in the case of the trivial level. The map (1.3) sends holomorphic half-integral weight forms of weight $r + 1/2$ to holomorphic modular forms of even integral weight $2r$.

The question as to whether there is any connection between the Borcherds isomorphism and the Kohnen-Shimura map is asked in Borcherds' paper [3, Problem 17.10]. We address this question in the framework of Serre's theory of p -adic modular forms. The point is to construct p -adic modular forms of half-integral weight from nearly holomorphic forms in $\mathcal{M}_{r+1/2}^+(\Gamma_0(4))$. We also extend the Kohnen-Shimura map $\mathcal{L}_{\delta,r}$ to p -adic modular forms. In this setting, if $r \in X = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{Z}_p^*)$, then $\mathcal{L}_{\delta,r}$ maps a p -adic modular form of weight $r + 1/2$ to one of weight $2r$.

We use our construction to produce p -adic modular forms out of f_d and apply our extension of the Kohnen-Shimura map to such forms. After a certain modification we obtain p -adic modular forms of weight 2. On the other side, we apply a result of Bruinier and Ono [6] and produce p -adic modular forms of weight 2 out of \mathcal{H}_d (and modify these forms slightly). The coincidence of the p -adic modular forms of weight 2 obtained by the two different ways provides an answer to the question by Borcherds mentioned above (see Theorems 2.10 and 2.12 below).

The paper also contains other applications of Theorem 2.1. These are congruences for Zagier's traces of singular moduli and p -adic limit formulas for class numbers of imaginary quadratic fields. These results allow us to partially answer the questions of Ono [12, Problems 4.31 and 7.30] (see Theorem 2.3 and its corollaries below). Recently Boylan [4] answered Ono's question [12, Problem 7.30] for the prime $p = 2$ using different methods.

In Section 2, we precisely state the main theorems mentioned above and their corollaries, and in Section 3 we prove the main theorems. The important tools involved in the proofs are Serre’s theory of p -adic modular forms [14, 15], Kohnen’s refinement of Shimura’s correspondence [10], and a version of Hida’s control theorem [8, Chapter 7].

Although both our method and our setting are different, we want to mention the connection between our Theorem 2.1 and the recent research of Ahlgren and Ono (see [1, 2]) which concerns congruence properties of the partition function and the traces of singular moduli $t(d)$. In both cases certain half-integral weight nonholomorphic modular forms are proved to be p -adic modular forms.

2 Results

Let p be an odd prime. We embed \mathbb{Z} into \mathbb{Z}_p and identify rational integers with their images under this embedding. Following Serre [14], we define a p -adic modular form as a formal power series

$$\phi = \sum_{n \geq 0} a(n)q^n \in \mathbb{Z}_p[[q]] \tag{2.1}$$

such that for any positive integer A there exists a true (complex-analytic) modular form $F = F_A$ on the full modular group $SL_2(\mathbb{Z})$ of some even integer weight $k(A)$ with rational integer Fourier coefficients with

$$F_A - \phi \equiv 0 \pmod{p^A} \tag{2.2}$$

coefficientwise. According to [14, Théorème 2], the sequence $\{k(A)\}$ has a limit as $A \rightarrow \infty$, which is an even (see [14, Section 1.4]) element of the group

$$X = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{Z}_p^*) = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z} \tag{2.3}$$

of continuous p -adic characters of \mathbb{Z}_p . This limit is called the weight of the p -adic modular form ϕ .

We tacitly identify rational integers with their images in X under the map $k \mapsto \{z \mapsto z^k\}$ for any $z \in \mathbb{Z}_p$ and $k \in \mathbb{Z}$ and adopt the exponential notation z^x for the image of $z \in \mathbb{Z}_p$ under $\chi \in X$. This is a slight abuse of notation since $z^x = 0$ if $\chi \in X$ for any $z \in \mathbb{Z}_p \setminus \mathbb{Z}_p^*$, but $z^x \neq 0$ if $\chi \in \mathbb{Z}$. We call an element $r \in X$ *even* if $r \in 2X$ and *odd* otherwise.

We extend the above definition to modular forms of half-integral weight in the obvious way: we require the weights $k(A)$, which we now write as $r(A) + 1/2$, to be half-integers. It is a result of Kohnen (see [9, Theorem 2A]) that an analogue of [14, Théorème 2] holds in this case. Namely, the sequence $\{r(A)\}$ has a limit as $A \rightarrow \infty$, which is an element $r \in X$ (not necessarily even in this case). We call $r + 1/2$ the weight of a p -adic modular form. Notice that our p -adic modular forms of half-integral weight automatically satisfy Kohnen’s plus-condition (for $r \in X$ we assume that $(-1)^r = \pm 1$ depending on the parity of r). Denote by $M_x^{(p)}$ the space of p -adic modular forms of integral (half-integral) weight $x \in X$ (resp., $x \in 1/2 + X$).

We construct p -adic modular forms associated with nearly holomorphic modular forms of half-integral weight. For a formal q -series

$$\Phi = \sum_{n \gg -\infty} a(n)q^n \tag{2.4}$$

and $\epsilon = \pm 1$ or 0 define the series without principal part

$$\Phi^\epsilon = \sum_{(n/p)=\epsilon, n \geq 0} a(n)q^n. \tag{2.5}$$

Theorem 2.1. For a positive integer m let

$$G = q^{-m} + \sum_{n \geq 0} a(n)q^n \in M_{r+1/2}^+(\Gamma_0(4)). \tag{2.6}$$

- (a) If $(m, p) = 1$, then $G^\epsilon, G^0 \in M_{r+1/2}^{(p)}$, where $\epsilon = -(\frac{-m}{p})$.
- (b) If $p \mid m$, then $G^{\pm 1} \in M_{r+1/2}^{(p)}$. □

Remark 2.2. If $(p, m) = 1$, then using the usual U_p and V_p operators and twists,

$$\begin{aligned} G^0 &= G|U_p|V_p, \\ G^\epsilon &= \frac{1}{2} \left(G - \left(\frac{-m}{p} \right) G \otimes \left(\frac{\cdot}{p} \right) - G^0 \right). \end{aligned} \tag{2.7}$$

Note that Theorem 2.1 is closely connected to the result of Serre [15, Théorème 5.2], where certain p -adic modular forms of weight 0 are associated with the modular invariant j . We postpone the proof of Theorem 2.1 to Section 3.

In [16] Zagier considers the traces

$$t(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{1}{w_Q} (j(\alpha_Q) - 744) \tag{2.8}$$

and proves that the generating function for the numbers $t(d)$

$$g_1 = q^{-1} - 2 - \sum_{d>0} t(d)q^d \in \mathcal{M}_{3/2}(\Gamma_0(4)) \tag{2.9}$$

is a nearly holomorphic modular form of weight $3/2$ (defined uniquely by the principal part and the constant term of its q -expansion), and moreover $t(d) = A(1, d)$.

Numerical examples. (1) Theorem 2.1, applied to the modular form g_1 , implies the existence of many congruences for the traces of singular moduli $t(d)$. In particular, for $p = 3$ we have

$$\begin{aligned} g_1^0 + 2g_1^1 &\equiv 1 + \sum_{d>0} \left(\left(\frac{-d}{3} \right) - 1 \right) t(d)q^d \\ &\equiv 1 - \sum_{3|d>0} t(d)q^d \\ &\equiv 1 + 2 \sum_{n>0} q^{3n^2} \equiv \theta(\tau)^3 \equiv 18H_{7/2} \pmod{3}, \end{aligned} \tag{2.10}$$

where $\theta(\tau) = 1 + 2 \sum_{n>0} q^{n^2}$ is the usual θ -function and $H_{7/2}$ is the Cohen’s Eisenstein series of weight $7/2$. The latter is the Clausen-von Staudt congruence for generalized Bernoulli numbers proven in [13]. In other words,

$$t(d) \equiv \begin{cases} 1 & \text{if } d = 3n^2 \\ 0 & \text{if } d \equiv 1 \pmod{3} \\ 0 & \text{if } \frac{d}{3} \in \mathbb{Z} \text{ is not a perfect square} \end{cases} \pmod{3}. \tag{2.11}$$

(2) One can apply Theorem 2.1 to the nearly holomorphic modular forms f_d and prove congruences for the associated Borcherds exponents. In particular, for $p = 5$ and $d = 3$ one has

$$f_3^0 + 2f_3^1 \equiv \sum_{D \geq 1} \left(1 + \left(\frac{D}{5} \right) \right) A(D, d)q^D \equiv 4 \sum_{5 \nmid n > 0} q^{n^2} \pmod{5}. \tag{2.12}$$

Note that $\mathcal{H}_3 = j^{1/3} = E_4 \Delta^{-1/3}$, where $\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$, and denote by $c(n)$ the Borcherds exponents for the weight 4 Eisenstein series

$$E_4 = 1 + 240 \sum_{n \geq 1} \sum_{d|n} d^3 q^n = (1 - q)^{-240} (1 - q^2)^{26760} \dots = \prod_{n \geq 1} (1 - q^n)^{c(n)}. \tag{2.13}$$

Then the above congruence translates into

$$c(n) \equiv \begin{cases} 3 & \text{if } 5 \mid n \\ 0 & \text{otherwise} \end{cases} \pmod{5}. \quad (2.14)$$

As another application, Theorem 2.1 provides in certain cases a p -adic modular form (of half-integral weight) with a nonzero constant term. This allows us to make use of the ideas involved in [14, Théorème 7] and to obtain limit formulas for class numbers of imaginary quadratic fields. For a negative fundamental discriminant $\delta < -4$ denote by $h(\delta)$ the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{\delta})$. Denote by $S_2(\Gamma_0(p))$ the space of cusp forms of weight 2 with respect to the congruence subgroup $\Gamma_0(p)$ and with trivial Nebentypus, and by T_l for a prime l the usual Hecke operator acting on this space (see (3.17) below).

Theorem 2.3. Let $\psi_1, \dots, \psi_{w_p}$ be the basis of $S_2(\Gamma_0(p))$ which consists of normalized Hecke eigenforms. Suppose that there is a finite set of primes P with the property that for every ψ_i , there is an $l \in P$ such that $\psi_i \mid T_l = 0$. If $L = \prod_{l \in P} l$, then the following p -adic limit holds:

$$\left(1 - \left(\frac{\delta}{p}\right)\right) h(\delta) = \frac{p-1}{24} \lim_{n \rightarrow \infty} \sum_{v \mid L} \left(\frac{\delta}{v}\right) t\left(p^{2n} \frac{L^2}{v^2} |\delta|\right). \quad (2.15) \quad \square$$

Note that $p \nmid L$ by [11, Theorem 4.6.17(2)].

If we remove the hypothesis stated in the theorem on the existence of the set of primes P , then we obtain a general, more complicated p -adic limit formula. In particular, Theorem 2.3 is a special case of this result chosen for esthetics (see the discussion of (3.25) below).

In particular, the hypotheses of Theorem 2.3 are satisfied when $p \in \{3, 5, 7, 13\}$, in which case $\dim S_2(\Gamma_0(p)) = 0$, and when $p \in \{11, 17, 19\}$, in which case $\dim S_2(\Gamma_0(p)) = 1$. For these p , Theorem 2.3 becomes as follows.

Corollary 2.4. (a) Let $p \in \{3, 5, 7, 13\}$. Then the following identity holds with a p -adic limit on the right:

$$\left(1 - \left(\frac{\delta}{p}\right)\right) h(\delta) = \frac{p-1}{24} \lim_{n \rightarrow \infty} t(p^{2n} |\delta|). \quad (2.16)$$

(b) Let $p \in \{11, 17, 19\}$ and assume that the corresponding modular elliptic curve $X_0(p)$ has supersingular reduction at a prime l . Then

$$\left(1 - \left(\frac{\delta}{p}\right)\right)h(\delta) = \frac{p-1}{24} \lim_{n \rightarrow \infty} \left(\left(\frac{\delta}{l}\right)t(p^{2n}|\delta|) + t(p^{2n}l^2|\delta|) \right). \tag{2.17}$$

□

Remark 2.5. If $\delta = -3$ (resp., $\delta = -4$), Corollary 2.4 remains true, but one has to multiply the right-hand side by 3 (resp., by 2). A slight modification of our argument may allow to obtain a similar result for $p = 2$.

Part (a) of Corollary 2.4 is very similar to [5, Theorem 9] and [6, Corollary 3]. This similarity is not, however, a direct connection: the current author cannot derive Corollary 2.4 from [6] or [5], nor can he derive the results of [6] or [5] from Corollary 2.4. If the prime p does not split in $\mathbb{Q}(\sqrt{\delta})$, then part (b) of Corollary 2.4 illustrates how Theorem 2.3 partially answers the question of Ono [12, Problem 4.31] about more general (i.e., $p > 7$) limit formulas for class numbers similar to those found in [5, 6].

Assume now that the prime p splits in $\mathbb{Q}(\sqrt{\delta})$. In this case the congruence

$$t(p^2|\delta|) \equiv 0 \pmod{p} \tag{2.18}$$

is proven in [6, Theorem 1.1] and [12, Theorem 7.14]. The question about a natural generalization of this congruence modulo higher powers of p is posed by Ono in [12, Problem 7.30]. Note that in this case the left-hand side of (2.15) is zero, and under the assumptions of Theorem 2.3 we obtain the following generalization of (2.18) which asserts that certain linear combinations of the numbers $t(p^{2n}m^2|\delta|)$ are divisible by an arbitrary high power of p as n grows.

Corollary 2.6. (a) Under the assumptions of Theorem 2.3, let p split in $\mathbb{Q}(\sqrt{\delta})$. Then

$$\lim_{n \rightarrow \infty} \sum_{v|L} \left(\frac{\delta}{v}\right) t\left(p^{2n} \frac{L^2}{v^2} |\delta|\right) = 0. \tag{2.19}$$

(b) In particular, let $p \in \{3, 5, 7, 13\}$ and let $\delta < -4$ be a fundamental discriminant such that p splits in $\mathbb{Q}(\sqrt{\delta})$. Then

$$\lim_{n \rightarrow \infty} t(p^{2n}|\delta|) = 0. \tag{2.20}$$

□

Remark 2.7. Recently, Boylan [4] obtained a precise quantitative version of part (b) in the case $p = 2$.

Consider now $\mathcal{L}_{\delta,r}$ as a map on formal power series over \mathbb{Q}_p and extend formally its definition (1.3) for any $r \in X$ and a fundamental discriminant δ with $(-1)^r \delta > 0$. Defined in this way, $\mathcal{L}_{\delta,r}$ takes p -adic modular forms of half-integral weight $r + 1/2$ to p -adic modular forms of integral weight $2r$ for any $r \in X$ (including the case $r = 0$ and $\delta = 1$). We apply this map to the p -adic modular forms of weight $1/2$ obtained in Theorem 2.1 from the modular forms f_d defined in (1.2). Notice that our extended definition of $\mathcal{L}_{\delta,r}$ imposes the condition $(m, p) = 1$ on the inner sum of (1.3). Also, since many Fourier coefficients of f_d^ϵ are zero, we have to impose certain conditions on δ and ϵ so that $\mathcal{L}_{\delta,0}(f_d^\epsilon)$ is not a priori identically zero. For instance, if $\epsilon = \pm 1$, then $(\frac{\delta}{p}) = \epsilon$. In particular, $(p, d) = 1$ in this case implies $(\frac{-d\delta}{p}) = -1$. In this way we obtain p -adic modular forms of weight 0 which are intimately connected to the logarithm of (1.1).

Theorem 2.8. Let $\delta > 0$ be a fundamental discriminant and let p be a prime.

(a) If $p \nmid \delta$, then put $\epsilon = (\frac{\delta}{p})$. If $p \nmid d$, then assume that $(\frac{\delta}{p}) = -(\frac{-d}{p})$. It holds that

$$\mathcal{L}_{\delta,0}(f_d^\epsilon) = \sum_{\substack{n>0 \\ (p,n)=1}} \sum_{m|n} \left(\frac{\delta}{m}\right) m^{-1} A\left(\frac{n^2}{m^2}\delta, d\right) q^n \in M_0^{(p)}. \tag{2.21}$$

(b) If $\epsilon = 0$, then $\mathcal{L}_{\delta,0}(f_d^0) \in M_0^{(p)}$, where

$$\mathcal{L}_{\delta,0}(f_d^0) = \begin{cases} \sum_{n>0} \sum_{m|n} \left(\frac{\delta}{m}\right) m^{-1} A\left(\frac{n^2}{m^2}\delta, d\right) q^n & \text{if } p \mid \delta, \\ \sum_{\substack{n>0 \\ p|n}} \sum_{\substack{m|n \\ (p,m)=1}} \left(\frac{\delta}{m}\right) m^{-1} A\left(\frac{n^2}{m^2}\delta, d\right) q^n & \text{if } p \nmid \delta. \end{cases} \tag{2.22}$$

□

Following [14] we define the operator R_h acting on the formal power series in $\mathbb{Z}_p[[q]]$ by

$$\sum_n c(n)q^n \mid R_h := \sum_{(n,p)=1} n^h c(n)q^n. \tag{2.23}$$

It is a result of Serre (see [14, Théorème 5]) that if $f \in M_w^{(p)}$ with an even $w \in X$, then $f \mid R_h \in M_{w+2h}^{(p)}$. Similarly (see Lemma 3.2 below), the operator R_h acts on p -adic modular forms of half-integral weight, again shifting the weight by $2h$. Moreover, it is easy to see that the commutation relation $\mathcal{L}_{\delta,r}(f) \mid R_{2h} = \mathcal{L}_{\delta,r+2h}(f \mid R_h)$ holds.

Consider now the following data: a positive integer d which is congruent to 0 or 3 modulo 4 and a prime p which is inert or ramified in $\mathbb{Q}(\sqrt{-d})$. The main result of [6] makes it possible to associate to this data a p -adic modular form of weight 2.

Theorem 2.9 (Bruinier-Ono). The logarithmic derivative of \mathcal{H}_d divided by $2\pi i$,

$$F_d := \frac{1}{2\pi i} \frac{1}{\mathcal{H}_d} \frac{d}{d\tau} \mathcal{H}_d(\tau) = -H(d) - \sum_{n>0} \sum_{m|n} m A(m^2, d) q^n, \tag{2.24}$$

is a p -adic modular form of weight 2. □

We now put $\delta = 1$ in Theorem 2.8. If p splits in $\mathbb{Q}(\sqrt{-d})$, then Theorem 2.1 implies that $\epsilon = -(\frac{-d}{p}) = -1$ or $\epsilon = 0$. However, $\mathcal{L}_{1,0}(f_d^{-1}) = 0$, and $\mathcal{L}_{1,0}(f_d^0) | R_1 = 0$. Thus we have to assume that $(\frac{-d}{p}) = -1$ or 0, which is exactly the condition under which Theorem 2.9 holds. Combining (2.21), (2.24), and (2.23) we obtain the following.

Theorem 2.10. Let $\epsilon = 1$ if $(\frac{-d}{p}) = -1$ and $\epsilon = \pm 1$ if $p \mid d$. Then the following equality of p -adic modular forms of weight 2 holds:

$$\mathcal{L}_{1,0}(f_d^\epsilon) | R_1 = F_d | R_0. \tag{2.25}$$

□

One may also consider the Kohnen-Shimura map $\mathcal{L}_{\delta,0}$ for a positive fundamental discriminant δ , not necessarily just $\delta = 1$. Let D and $-d$ be coprime positive and negative fundamental discriminants. Then the genus character $\chi_{D,d}$ assigns to any quadratic form Q of discriminant $-dD$ a value ± 1 defined by $\chi_{D,d}(Q) = (\frac{D}{l}) = (\frac{-d}{l})$, where l is any prime represented by Q and not dividing Dd . (This is independent of the choice of l .) Following [16] consider the weight 0 modular form

$$\mathcal{H}_{D,d} = \prod_{Q \in \mathcal{Q}_{D,d}/\Gamma} (j(\tau) - j(\alpha_Q))^{\chi_{D,d}(Q)} \tag{2.26}$$

and note as in [16] that the q -expansion of $\log \mathcal{H}_{D,d}$, which begins with 1, belongs to $\sqrt{D}\mathbb{Q}[[q]]$. The result of [6] applies to this case and implies the following analogue of Theorem 2.9.

Theorem 2.11. If $(\frac{-dD}{p}) = -1$ or 0, then the logarithmic derivative of $\mathcal{H}_{D,d}$ divided by $2\pi i\sqrt{D}$,

$$F_{D,d} := \frac{1}{2\pi i\sqrt{D}} \frac{1}{\mathcal{H}_{D,d}} \frac{d}{d\tau} \mathcal{H}_{D,d}(\tau), \tag{2.27}$$

is a p -adic modular form of weight 2. □

We now put $\delta = D$ and combine Theorems 2.8 and 2.11 and (2.23) with the generalization of the Borcherds product expansion given in [16, Theorem 7] to deduce the following generalization of Theorem 2.10.

Theorem 2.12. Put

$$\epsilon = \begin{cases} 1 & \text{if } \left(\frac{-dD}{p}\right) = -1, \\ \pm 1 & \text{if } p \mid d, (p, D) = 1, \\ 0 & \text{if } p \mid D, (p, d) = 1. \end{cases} \tag{2.28}$$

Then the following equality of p -adic modular forms of weight 2 holds:

$$\mathcal{L}_{D,0}(f_d^\epsilon) \mid R_1 = F_{D,d} \mid R_0. \tag{2.29}$$

□

Note that Theorems 6 and 7 in [16] are stated there without a complete proof.

3 Proofs of Theorems 2.1 and 2.3

Let $M_{r+1/2}^+ = M_{r+1/2}^+(\Gamma_0(4))$ be the space of holomorphic modular forms of half-integral weight $r + 1/2$ with respect to $\Gamma_0(4)$ which satisfy Kohnen’s plus-condition.

We begin with the following half-integral weight analogue of [14, Théorème 5a].

Lemma 3.1. If $f = \sum a(n)q^n \in M_{r+1/2}^{(p)}$, then

$$\vartheta f = \frac{1}{2\pi i} \frac{df}{d\tau} = q \frac{df}{dq} = \sum na(n)q^n \in M_{r+2+1/2}^{(p)}. \tag{3.1}$$

□

Proof. We use the classical notation of Ramanujan

$$P(\tau) = 1 - 24 \sum_{n \geq 1} \sum_{m \mid n} mq^n. \tag{3.2}$$

It is well known (see [14, page 211]) that $P \in M_2^{(p)}$. One concludes that $P(4\tau)$ is a p -adic modular form of weight 2 on $\Gamma_0(4)$. The transformation law

$$P\left(\frac{a\tau + b}{c\tau + d}\right) = P(\tau)(c\tau + d)^2 - \frac{12c}{2\pi i}(c\tau + d) \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \tag{3.3}$$

implies that the differential operator

$$\vartheta - \frac{2t + 1}{6}P(4\tau) \tag{3.4}$$

takes elements of $M_{t+1/2}^+$ to elements of $M_{t+2+1/2}^+$. From this point on one can repeat Serre’s argument [14, page 211, proof of Théorème 5a], making the necessary changes in notations, in order to finish the proof of Lemma 3.1. ■

We will also need the following generalization of a result of Serre (see [14, Théorème 5]) to the half-integral weight case.

Lemma 3.2. Let $f = \sum_{n \geq 0} c(n)q^n \in M_{w+1/2}^{(p)}$, and $w, h \in X$. Then

$$f | R_h = \sum_{(n,p)=1} n^h c(n)q^n \in M_{w+2h+1/2}^{(p)}. \tag{3.5}$$

Proof. The assertion follows from Lemma 3.1 in the same way as [14, Théorème 5b] follows from [14, Théorème 5a]. Indeed, choose a sequence of positive integers h_i such that $h_i \rightarrow h$ in X and $|h_i| \rightarrow \infty$. By Lemma 3.1, $\vartheta^{h_i} f \in M_{w+2h_i+1/2}^{(p)}$. The assertion of the lemma follows from $\lim_{i \rightarrow \infty} \vartheta^{h_i} f = f | R_h$. ■

We use the standard notation $\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$ for the unique cusp form of weight 12.

Lemma 3.3. If m and r are positive integers and

$$f = \sum_{n \geq m} b(n)q^n \in M_{r+1/2}^+, \tag{3.6}$$

then $f(\tau)/\Delta(4\tau)^N \in M_{r-12N+1/2}^+$, where $N = \lceil m/4 \rceil$. ■

Proof. It follows from [10, Proposition 1] that there exist modular forms ϕ and ψ (of even integer weight, on $SL_2(\mathbb{Z})$) such that

$$f(\tau) = \begin{cases} \phi(4\tau)\theta(\tau) + \psi(4\tau)H_{5/2}(\tau) & \text{if } r \text{ is even,} \\ \phi(4\tau)H_{7/2}(\tau) + \psi(4\tau)H_{11/2}(\tau) & \text{if } r \text{ is odd.} \end{cases} \tag{3.7}$$

Here, $H_{t+1/2}$ denotes the Cohen’s Eisenstein series [7] of weight $t+1/2$, normalized so that the constant term is 1 and $\theta = \sum_{n \in \mathbb{Z}} q^{n^2}$. Since $H_{5/2}/\theta = 1 - 12q + \dots$ and $H_{11/2}/H_{7/2} = 1 - 144q^3 + \dots$, we have

$$\phi(4\tau) = \begin{cases} -\psi(4\tau)(1 - 12q + \dots) + O(q^m) & \text{for even } r, \\ -\psi(4\tau)(1 - 144q^3 + \dots) + O(q^m) & \text{for odd } r. \end{cases} \tag{3.8}$$

The function $\phi(4\tau)$ is invariant under $\tau \mapsto \tau + 1/4$, which implies that the q -expansion of $\psi(\tau)$ begins with $q^{\lceil m/4 \rceil}$. This implies that $\psi(\tau) = O(q^N)$. Therefore, since ψ is a modular form, it is divisible by Δ^N . A similar argument implies that $\phi(\tau)$ is divisible by Δ^N . Lemma 3.3 follows now from (3.7). ■

Recall that

$$\frac{\vartheta\Delta(4\tau)}{\Delta(4\tau)} = 4P(4\tau). \quad (3.9)$$

Lemma 3.4. For an integer $r \geq 0$ let $G \in \mathcal{M}_{r+1/2}^+(\Gamma_0(4))$ and let $M \geq 0$ be an integer such that $h(\tau) = \Delta(4\tau)^M G(\tau)$ is holomorphic. Then, for any integer $a \geq 1$,

$$\Delta(4\tau)^M \vartheta^a(G(\tau)) \in \mathcal{M}_{r+12M+2a+1/2}^{(p)}. \quad (3.10)$$

□

Proof. Differentiating $h(\tau) = \Delta(4\tau)^M G(\tau)$ using (3.9), we obtain

$$\Delta(4\tau)^M \vartheta(G(\tau)) = \vartheta(h)(\tau) - 4Mh(\tau)P(4\tau). \quad (3.11)$$

The right-hand side belongs to $\mathcal{M}_{r+12M+2+1/2}^{(p)}$ by Lemma 3.1 and the fact that $P(4\tau)$ is a p -adic modular form (of weight 2 on $\Gamma_0(4)$). Therefore so is the left-hand side. We did not use the fact that $h \in \mathcal{M}_{r+12M+1/2}^+$ since it was sufficient to have $h \in \mathcal{M}_{r+12M+1/2}^{(p)}$ instead. After this observation an induction in a finishes the proof of Lemma 3.4. ■

Proof of Theorem 2.1. We keep the notations of Theorem 2.1; in particular, $m > 0$ is an integer, $G = q^{-m} + \mathcal{O}(1) \in \mathcal{M}_{r+1/2}^+(\Gamma_0(4))$, and $\epsilon = -(\frac{-m}{p})$.

Let R be any positive integer. For a positive integer l put $a = (p-1)p^l/2$. Let $M \geq [m/4] + 1$ be an integer. By Lemma 3.4,

$$h_l := \begin{cases} \Delta(4\tau)^M (G(\tau) + \epsilon \vartheta^a G(\tau)) & \text{if } (m, p) = 1, \\ \Delta(4\tau)^M \vartheta^t G(\tau) & \text{with } t = a \text{ or } 2a \text{ if } m \equiv 0 \pmod{p}, \end{cases} \quad (3.12)$$

is a p -adic modular form of half-integral weight. The two equations in (3.12) above correspond to the two parts of Theorem 2.1. Thus

$$h_l = \begin{cases} \Delta(4\tau)^M \left(\frac{1 + (-m)^a \epsilon}{q^m} + \sum_{n \geq 0} (1 + \epsilon n^a) a(n) q^n \right) & \text{if } (m, p) = 1, \\ \Delta(4\tau)^M \left(\frac{(-m)^t}{q^m} + \sum_{n \geq 0} n^t a(n) q^n \right) & \text{with } t = a \text{ or } 2a \text{ if } p \mid m, \end{cases} \quad (3.13)$$

and since $1 + (-m)^a \epsilon \equiv (-m)^t \equiv 0 \pmod{p^l}$, for any $r \geq 0$ there exists a holomorphic half-integral weight modular form $H_{l,r}$ with

$$H_{l,r} = p^l \sum_{n=0}^{4M-1} b_{l,r}(n)q^n + \sum_{n \geq 4M} b_{l,r}(n)q^n \equiv h_l \pmod{p^r} \tag{3.14}$$

with integer coefficients $b_{l,r}(n)$. We now fix a sufficiently big r and vary l such that $R \leq l \leq r$. Note that all h_l , and therefore all $H_{l,R}$, coincide modulo p^R . There is a nontrivial linear combination F_R of $H_{l,R}$ such that $F_R \equiv h_R \pmod{p^R}$ and $F_R = O(q^{4M})$. In view of [9, Theorem 2A] and the congruence $E_{p-1} = 1 - (2k/B_k) \sum_{n \geq 1} \sigma_{p-2}(n)q^n \equiv 1 \pmod{p}$ (see [14, Section 1.1.d] for the details), we may assume that the modular forms $H_{l,R}$ involved in this linear combination have equal weights. It follows now from Lemma 3.3 that $F_R(\tau)/\Delta(4\tau)^M$ is a holomorphic modular form of half-integral weight. By construction,

$$\frac{F_R(\tau)}{\Delta(4\tau)^M} \equiv \begin{cases} G(\tau) + \epsilon \vartheta^{(p-1)p^R/2} G(\tau) & \text{if } (m, p) = 1 \\ \vartheta^{(p-1)p^R/2} G(\tau) & \text{if } m \equiv 0 \pmod{p} \text{ (} t = a \text{) } \pmod{p^R}, \\ \vartheta^{(p-1)p^R} G(\tau) & \text{if } m \equiv 0 \pmod{p} \text{ (} t = 2a \text{)} \end{cases} \tag{3.15}$$

with an arbitrary positive R , which was chosen at the beginning. Since $(n/p) \equiv n^{((p-1)/2)p^R} \pmod{p^R}$, the congruences (3.15) imply that the series

$$\begin{aligned} 2G^\epsilon + G^0 &= \sum_{n \geq 0} \left(1 - \left(\frac{-m}{p} \right) \left(\frac{n}{p} \right) \right) a(n)q^n \quad \text{if } (m, p) = 1, \\ G^{+1} - G^{-1} &= \sum_{n \geq 0} \left(\frac{n}{p} \right) a(n)q^n \quad \text{if } m \equiv 0 \pmod{p}, \\ G^{+1} + G^{-1} &= \sum_{(n,p)=1} a(n)q^n \quad \text{if } m \equiv 0 \pmod{p}, \end{aligned} \tag{3.16}$$

are p -adic modular forms of half-integral weight. Finally, we apply Lemma 3.2 to the sum $2G^\epsilon + G^0$ to conclude that its separate summands are p -adic modular forms. (Indeed, pick $h = 0 \in X$ in Lemma 3.2 and notice that $G^{\pm 1} | R_0 = G^{\pm 1}$ and $G^0 | R_0 = 0$.) Theorem 2.1 is now proved. ■

Proof of Theorem 2.3. Recall the standard definition of the (weight 2) Hecke operator T_l for a prime l :

$$\sum_{n \geq 0} c(n)q^n | T_l = \sum_{n \geq 0} \left(c(ln) + lc \left(\frac{n}{l} \right) \right) q^n. \tag{3.17}$$

It follows from Theorem 2.1 that

$$g_1^0 = 2 + \sum_{n \equiv 0 \pmod p} t(n)q^n \in M_{3/2}^{(p)}. \tag{3.18}$$

In other words, there is a sequence of holomorphic modular forms $F_r \in M_{r+1/2}$ with rational integer coefficients such that $F_r \rightarrow g^0$ coefficientwise p -adically as $r \rightarrow 1$ in X and $r \rightarrow \infty$ as integers. Apply the Kohlen-Shimura map $\mathcal{L}_{\delta,r}$ to F_r to obtain modular forms

$$\Phi_r = \sum_{n \geq 0} a(n)q^n \in M_{2r} \tag{3.19}$$

with

$$\begin{aligned} a(0) &= L\left(1-r, \left(\frac{\delta}{\cdot}\right)\right), \\ a(n) &= \sum_{m|n} \left(\frac{\delta}{m}\right) m^{r-1} t\left(\frac{n^2}{m^2}|\delta|\right). \end{aligned} \tag{3.20}$$

For a positive integer u the series

$$\Phi_{r,u} = \sum_{n \geq 0} a(p^u n)q^n \in M_{2r}(\Gamma_0(p)) \tag{3.21}$$

is the q -expansion of a modular form of weight $2r$ with respect to $\Gamma_0(p)$. Consider Hida’s ordinary projector $\mathcal{E} = \lim_{m \rightarrow \infty} U_p^{(p-1)p^m}$. If u is big enough, then the power series $\Phi_{r,u}$ and $\mathcal{E}(\Phi_{r,u})$ in $\mathbb{Z}_p[[q]]$ are congruent modulo a high power of p . According to [8, Chapter 7], $\mathcal{E}(\Phi_{r,u})$ is a specialization at weight $2r$ of a p -ordinary Λ -adic form. Denote by $\phi_{r,u}$ the specialization of this Λ -adic form at weight 2. As r approaches 1 in X , the q -expansions of $\mathcal{E}(\Phi_{r,u})$ and $\phi_{r,u}$ become congruent modulo arbitrary high powers of p . Thus, if both integers r and u are big enough, then $\Phi_{r,u}$ and $\phi_{r,u}$ are p -adically close.

Formula (3.20) imply that if r and u are big enough, then $\phi_{r,u}$ is congruent modulo an arbitrary high power of p to

$$\left(1 - \left(\frac{\delta}{p}\right)\right)h(\delta) + \sum_{n \geq 1} q^n \sum_{v|n(p,v)=1} \left(\frac{\delta}{v}\right) t\left(p^{2u} \frac{n^2}{v^2}|\delta|\right). \tag{3.22}$$

According to a version of the Hida control theorem [8, Theorem 7.3],

$$\phi_{r,u} = \alpha_0(r,u)P^* + \sum_{i=1}^{w_p} \alpha_i(r,u)\psi_i, \tag{3.23}$$

where

$$P^* = P(\tau) - pP(p\tau) = 1 - p - 24 \sum_{n \geq 1} \left(\sum_{m|n} m - p \sum_{m|n/p} m \right) q^n \in M_2(\Gamma_0(p)) \quad (3.24)$$

is the p -ordinary weight 2 Eisenstein series with respect to $\Gamma_0(p)$. The coefficients $\alpha_i(r, u)$ belong to the p -adic completion of a finite extension of \mathbb{Q} . Now apply the weight 2 Hecke operators T_{l_j} to the identity (3.23) to eliminate the cusp forms ψ_i (recall that $\psi_j | T_{l_j} = 0$ by the assumption of Theorem 2.3).

Note that a similar but more complicated result may be obtained making use of Hecke operators T_l with $\psi_j | T_l = \lambda_j(l)\psi_j$. In this case one may use the operators

$$\phi_{r,u} | S_l = \phi_{r,u} | T_l - \lambda_j(l)\phi_{r,u} \quad (3.25)$$

in order to eliminate cusp forms from the linear combination (3.23).

We conclude that $\phi_{r,u} | T_L = \phi_{r,u} | T_{l_1} | \cdots | T_{l_w}$ is a nonzero (because the constant term does not vanish) multiple of P^* . If we denote by $A_0(r, u)$ and $A_1(r, u)$ the constant term and the first q -expansion coefficient of $\phi_{r,u} | T_L$, then we have

$$\frac{A_0(r, u)}{A_1(r, u)} = \frac{p-1}{24}. \quad (3.26)$$

We now combine this identity with (3.22) together with formula (3.17) for the action of Hecke operators of weight 2 and take the limit when $r \rightarrow 1$ in X and both r and u go to infinity as positive integers to obtain (2.15). ■

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