ON A CONJECTURE OF ATKIN

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Abstract. Let j be the modular invariant. For the primes $p \leq 23$ the q-expansion coefficients of $U^m(j-744)$ are multiplicative as it was a Hecke eigenform modulo a power of p which increases with m. This was conjectured by Atkin on the basis of extensive numerical experiments, and is proved in this paper. The cases $p = 5$, 7 and 11 are under special consideration in this paper.

1. Introduction

Let

$$j = q^{-1} + 744 + 196884q + 21493760q^2 + ... = q^{-1} + 744 + \sum_{n \geq 1} c(n)q^n$$

be the q-expansion of the modular j-function. Since j plays its role in such diverse issues of mathematics ranging from the theory of elliptic functions and elliptic curves to the theory of the Monster group, the rational integer coefficients $c(n)$ are interesting for many different reasons. For instance, the divisibility properties of these numbers were studied extensively in the connection with classical Ramanujan congruences. In particular, Lehner [17, Theorem 2] and Atkin [2, Theorem 1] proved that for positive integers m and n

$$c(np^m) \equiv \begin{cases} 0 \bmod p^{m+1} & \text{if } p = 5 \\ 0 \bmod p^m & \text{if } p = 7 \\ 0 \bmod p^m & \text{if } p = 11. \end{cases} \tag{1}$$

For a rational prime p the action of the Atkin $U_p = U$-operator on formal power series in variable q is defined by $f \mapsto U(f)$, where

$$f = \sum a(n)q^n \quad \text{and} \quad U(f) = \sum a(pn)q^n. \tag{2}$$

Put $t_m(n) := c(p^m n)/c(p^m)$. For a prime $p \leq 23$ (see (14) below for $p = 5$, 7 and 11) the (rational) numbers $t_m(n)$ are p-integral. Consider

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the normalized series

$$\frac{1}{c(p^m)}U^m(j-744) = \sum_{n\geq 1} t_m(n)q^n. \tag{3}$$

On the basis of extensive numerical evidence Atkin [3, 4] conjectured that this series behaves like a normalized Hecke eigenform modulo an increasing power of p when m is increasing: *"...repeated application of U leads to a series of Fourier coefficients which become multiplicative modulo increasing powers of p"*, [4]. We refer to this statement as Atkin's conjecture and prove it in this paper for $p = 5$, 7 or 11.

Theorem 1. *Let $p = 5$, 7 or 11. Put*

$$\mu = \begin{cases} 4 & \text{if } p = 5 \\ 2 & \text{if } p = 7 \\ 2 & \text{if } p = 11. \end{cases}$$

The congruences

$$t_m(nl) - t_m(n)t_m(l) + t_m(n/l)/l \equiv 0 \bmod p^{(\mu-1)m-C}, \tag{4}$$

and

$$t_m(np) - t_m(n)t_m(p) \equiv 0 \bmod p^{(\mu-1)m-C} \tag{5}$$

hold for every prime $l \neq p$, positive integer n, all $m \gg 0$ and some constant $C = C(p)$.

Numerical experiments do not suggest that these congruences hold for $p > 23$. In this connection, Atkin wrote in [3]:

"... it is remarkable that all these primes should exhibit the same behaviour with regard to our multiplicative congruence properties. It thus seems possible that some entirely different method or theory may exist which would give an uniform proof of all these cases."

The method which we present here is based, roughly, on the fact that the dimension of the minimal slope space of p-adic cusp forms of weight zero is one for all primes $5 \leq p \leq 23$. This minimal slope is zero for $p = 13, 17$,19 and 23, and positive for $p = 5, 7$ and 11. This one-dimensionality is the shared property of the primes under consideration which implies the congruences, because the properly normilized repeated application of U becomes a projection to a one-dimensional space, and the image, if non-zero, must be a Hecke eigenform. We thus provide a "uniform proof" which Atkin called for. This also explains why one does not expect similar congruences to hold for $p > 23$. We, however, have no proof that no similar congruences show up for all bigger primes. Nevertheless, our approach sheds some light on Atkin's observations concerning the primes $p = 29, 31$ and 37 in [3]. Indeed, on the basis of his numerical experiments Atkin suggested that the series (3) behaves like a linear combination of two (for $p = 29, 31$) or three (for $p = 37$) Hecke eigenforms modulo p for $m \geq 1$. This checks with the fact that the minimal slope in these cases is zero, and the dimension of the slope zero subspace is two (for $p = 29, 31$) or three (for $p = 37$) by Hida's control theorem. We bound ourselves with this observation and refrain from stating any general theorem for $p > 23$ in this paper.

Although we concentrate here on the primes $p = 5, 7$ and 11, a slight modification of our argument allows to prove Theorem 1 with $\mu = 2$ for $p = 13, 17$,19 and 23, which is a quantification of an earlier qualitative result of the author [13]. Indeed, for the latter four primes Proposition 1 below remains true with $\mu = 1$ and $ord_p(\lambda_p) = 0$ (note that $(\mu - 1)$ in (4,5) is in fact $(\mu - ord_p(\lambda_p)))$. The proof is based on similar computations. An analogue of Proposition 2 below follows easily from (the proof of) [13, Corollary 1]. Since these two propositions are established, one derives Theorem 1 from them as we do it here *mutatis mutandis*. For the sake of space we omit the details, and do not consider the (easier) case of these primes in this paper.

Atkin's numerical experiments suggest (and he actually also conjectured) that congruences (4,5) with $\mu = 2,\ C = 0$ and $m \geq 1$ hold for any prime $p \leq 23$. Our result is weaker than this specific form of Atkin's conjecture, because our methods do not allow us to provide any information about the constant C and a lower bound for m such that the congruences hold. At the same time, our result for $p = 5$ is (asymptotically) stronger than that predicted by Atkin.

The cases $p = 2$ and 3 deserve a separate consideration. In both cases an application of general theory, although possible, is inconvenient, and specific methods may imply sharper results. Specifically, a proof of Atkin's conjecture and certain congruences beyond the original conjecture for $p = 2$ were found by Akiyama [1]. In the remaining case $p = 3$, we conjecture (based on the results of [7]) that Theorem 1 is true with $\mu = 5$.

In the case when $p = 13$ Atkin's conjecture (with $\mu = 2,\ C = 0$ and $m \geq 1$) was proved by Atkin himself. His proof has been published in [18, Chapter VI]. Another proof was later discovered by Koike [15] as an application of a profound result of Koike and a result of Atkin and O'Brien [5]; the latter is specific for the case $p = 13$. A proof for the case $p = 13$ which makes use of a result of Dwork [8] was presented by Katz [14]. The current author's proof of a qualitative version of the conjecture [13] for $p = 13, 17, 19$ and 23 is based on a combination of Hida's Control Theorem and Serre's p-adic theory of modular forms. Specifically, this proof is based on the observation that, for these primes, the space of p-ordinary p-adic cusp forms of weight zero is one-dimensional. This is not true for $p < 13$: this space is empty. However, a similar statement holds. We consider Katz' overconvergent cusp forms instead of Serre's p-adic cusp forms and the spectral theory of U-operator developed by Gouvêa, Mazur, Coleman and Wan as a natural substitute for Hida's Control Theorem. We prove (see Proposition 1 below) that the subspace of minimal slope for weight zero is one-dimensional if $p = 5$, 7 or 11. This minimal slope turns out to be 1. The proof of Theorem 1 is based on this observation, an estimate for the next smallest slope, an argument which is based on classical calculations of Rademacher [19] and Lehner [17] pertaining to the modular equations for the levels 5 and 7, and a similar result of Atkin [2] in the case $p = 11$. This argument and Atkin's result are specific for the j-function.

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2. Proof of Theorem 1.

Let $p \geq 5$ be a fixed rational prime, and let B be the ring of integers in a finite extension K of the field $\mathbb{Q}_p$ of p-adic rational numbers. For an integer $k \geq 0$ and an element $r \in B$ let $M_k(B, r)$ be the space of r-overconvergent p-adic modular forms of weight k, tame level 1, defined over B. For an overconvergent form $f \in M_k(B, r)$ we denote by $f(q)$ its q-expansion. The U-operator is defined as the one-sided inverse to the Frobenius operator. It acts on $M_k(B, r)$, and its action on q-expansions is given by (2).

If $0 < ord_p(r) < p/(p+1)$ then (see [21, Lemma 2.2]) the U-operator induces a completely continuous endomorphism of the p-adic Banach space $M_k(K, r) = M_k(B, r) \otimes K$. This observation allows to apply the p-adic spectral theory developed by Serre in [20]. In particular, the Fredholm determinant $P_k(t) := \det(1 - tU | M_k(K, r))$ is a p-adic entire function with coefficients in B. The series $P_k(t)$ factorizes over B in terms of slopes of its Newton polygon:

$$P_k(t) = \sum_{n\geq 0} a_n(k)t^n = \prod_{\alpha\in\mathbb{Q}_{\geq 0}} P_k(t)^{(\alpha)},$$

where $P_k(t)^{(\alpha)} \in B[t]$ is the slope α part of $P_k(t)$. For each non-negative rational number α we have the spectral U-equivariant decomposition

$$M_k(K,r) = M_k(K,r)^\alpha \oplus F_\alpha,$$

where $M_k(K,r)^\alpha$ denotes the generalized eigenspace of U whose eigenvalues have p-adic valuation α. The characteristic polynomial of U acting $M_k(K,r)^\alpha$ is $P_k(t)^{(\alpha)}$. For any element $g \in M_k(K,r)^\alpha$ there exists $C \geq 0$ such that $ord_p(U^m(g)) \geq \alpha m - C$, where the p-adic order is understood in the sense of modular topology on $M_k(K,r)$. We refer to [10] for the definitions and the discussion of the interrelations between the modular and q-expansion topologies.

Proposition 1. *Let $p = 5$, $p = 7$ or 11. Assume that $0 < ord_p(r) < p/(p+1)$. Then*

$$P_0(t) = (1-t)(1-\lambda_p t)P_0^{(\mu)}(t)\prod_{\alpha>\mu} P_0(t)^{(\alpha)}$$

with $ord_p(\lambda_p) = 1$, and

$$\mu = \begin{cases} 4 & \text{if } p = 5 \\ 2 & \text{if } p = 7 \\ 2 & \text{if } p = 11. \end{cases}$$

Proof. Following [7] we use Koike's p-adic version of the Eichler-Selberg trace formula [16] in order to compute the first few quantities $ord_p(a_n(0))$:

$$\begin{array}{ll} p=5 & 0,0,1,5,10,18,27,\ldots \\ p=7 & 0,0,1,3,6,12,\ldots \\ p=11 & 0,0,1,3,6,10,\ldots. \end{array} \tag{6}$$

This calculation implies, in particular, that

$$ord_p(a_1(0)) = 0, \qquad ord_p(a_2(0)) = 1, \qquad ord_p(a_3(0)) = \mu + 1,$$

with μ as in Proposition 1. In order to prove Proposition 1, it is now sufficient to show that

$$ord_p(a_n(0)) > \begin{cases} 4n-7 & \text{if } p = 5 \\ 2n-3 & \text{if } p = 7 \\ 2n-3 & \text{if } p = 11 \end{cases} \qquad \text{for } n \geq 4. \tag{7}$$

In order to prove (7) we make use of the uniform lower bound for Newton polygons obtained by Wan [21, Lemma 3.1]. Reducing Wan's result to our setting and using the well-known formulas for the dimensions of the spaces of modular forms on $SL(2,\mathbb{Z})$, we obtain the inequalities for $p=5$ and 7

$$ord_p(a_n(0)) \geq \begin{cases} n^2-2n+\frac{2}{3} > 4n-7 & \text{if } n \geq 5 \text{ for } p=5 \\ \frac{3}{4}n^2-\frac{7}{4}n+\frac{3}{4} > 2n-1 & \text{if } n \geq 4 \text{ for } p=7. \end{cases}$$

If $p=11$, write $k=\lfloor n/5 \rfloor$, and $r=n-5k$. In these notations the result of Wan reduces to

$$ord_{11}(a_n(0)) \geq \begin{cases} \frac{25}{2}k^2+5k+\frac{5}{6} & \text{if } r=0 \\ \frac{25}{2}k^2-\frac{1}{6} & \text{if } r=1 \\ \frac{25}{2}k^2+5k+\frac{1}{2} & \text{if } r=2 \\ \frac{25}{2}k^2+10k+\frac{4}{3} & \text{if } r=3 \\ \frac{25}{2}k^2+10k+\frac{13}{3} & \text{if } r=4, \end{cases}$$

which implies $ord_{11}(a_n(0)) > 2n-3$ for $n \geq 5$. These inequalities combined with the precalculated values $ord_p(a_i(0))$ (6) prove (7). □

The p-adic Hecke operators T_l for primes $l \neq p$ are defined by Katz in [14, Section 3.12]. These operators preserve $M_0(B,r)$ and their action on the q-expansions is given by the classical formulas (see [9, Chapter II.1]):

$$T_l\left(\sum_{n\geq 0} a_n q^n\right) = \sum_{n\geq 0} a_{ln}q^n + l^{k-1}\sum_{n\geq 0} a_n q^{ln} \tag{8}$$

Proposition 1 implies the direct U-equivariant splitting

$$M_0(K,r) = M_0(K,r)^{(0)} \oplus M_0(K,r)^{(1)} \oplus M_0(K,r)^{(\mu)} \oplus_{\alpha>\mu} M_0(K,r)^{(\alpha)} \tag{9}$$

with $M_0(K,r)^{(0)} \simeq K$ and a one-dimensional component $M_0(K,r)^{(1)}$. The various T_l commute with each other. For $k=0$, they all commute with U (see [14, Section 3.12]). It follows that the (unique up to multiplication by a constant) element from $M_0(K,r)^{(1)}$ is a simultaneous Hecke eigenform. In particular, for a non-zero element $F \in M_0(K,r)^{(1)}$ with the q-expansion

$$F(q) = \sum_{n>0} a(n,F)q^n$$

we can and will assume that $a(1,F) \neq 0$. Let $t(n,F) := a(n,F)/a(1,F)$. Then (8) implies

$$t(nl,F) - t(n,F)t(l,F) + t(n/l,F)/l = 0 \tag{10}$$

and

$$t(np,F) - t(n,F)t(p,F) = 0 \quad \text{with } t(p,F) = \lambda_p. \tag{11}$$

We now consider, following [11], the spectral decomposition induced by (9). Let $S_p = \{1, \mu, \dots\}$ be the set of positive slopes in (9). For $j \in S_p$ and an element $f \in M_0(K,r)$ such that the constant term of its q-expansion is zero, let $e_j(f) \in M_0(K,r)^{(j)}$ be its projection to the direct summand $M_0(K,r)^{(j)}$. For a real non-negative x we put as in [11]

$$e_{[x]}(f) = \sum_{j \le x} e_j(f).$$

The p-adic case of spectral theory is analogous to the usual complex case. In particular, (see [11, Proposition 1]) there exists $\varepsilon > 0$ such that

$$ord_p(U^m(f - e_{[x]}(f))) \ge m(x + \varepsilon) \qquad \text{for } m \gg 0, \tag{12}$$

where the p-adic order is understood in the sense of modular topology on $M_0(K,r)$.

Katz showed in [14, Section 3.13] that if $p \ne 2, 3$ and $0 < ord_p(r) < p/(p+1)$ then

$$U(j) \in \frac{1}{p} M_0(B,r) \subset M_0(K,r),$$

which allows to apply (12) to the case when $f = U(j - 744)/\lambda_p$. Let

$$F := e_1(f) = e_1(U(j-744)/\lambda_p) \in M_0(K,r)^{(1)}. \tag{13}$$

Now (12) with $x = \mu$ implies

$$ord_p(U^{m+1}(j-744)/\lambda_p - \lambda_p^m F - U^m(e_\mu(f))) > \mu m \qquad \text{for } m \gg 0.$$

Since there exist $C \ge 0$ such that $ord_p(U^m(e_\mu(f))) \ge \mu m - C$, we obtain

$$ord_p\left(\frac{U^{m+1}(j-744)}{\lambda_p^{m+1}} - F\right) > (\mu - 1)m - C \qquad \text{for } m \gg 0.$$

This observation combined with the identity

$$\left(\frac{1}{\lambda_p} U\right)^m (j - 744) = \lambda_p^{-m} \sum_{n \ge 1} c(np^m) q^n = \frac{c(p^m)}{\lambda_p^m} \sum_{n \ge 1} t_m(n) q^n,$$

(10), (11) and the fact that the modular topology on $M(K,r)$ is stronger than the q-expansion topology implies the congruences claimed in Theorem 1 (possibly, with a different choice of C) if the quantity

$$ord_p(c(p^m)/\lambda_p^m) = ord_p(c(p^m)) - m$$

is bounded independently on $m \geq 1$. We address this latter fact in the following proposition which accomplishes the proof of Theorem 1.

Proposition 2. *Atkin's and Lehner's congruences* (1) *are best possible, namely*

$$ord_p(c(p^m)) = \begin{cases} m+1 & \text{if } p=5 \\ m & \text{if } p=7 \\ m & \text{if } p=11. \end{cases} \tag{14}$$

Remark. Proposition 2 immediately implies $F = e_1(U(j-744))/\lambda_p \neq 0$.

Proof. This was proved by Atkin [2, Section 3.5] in the case $p = 11$. Thus we assume p to be either 5 or 7 for the rest of the proof. Let $q = \exp(2\pi i\tau)$ with $\Im(\tau) > 0$. Consider the modular functions of weight zero with respect to the congruence subgroup $\Gamma_0(p)$

$$\varphi(\tau) = q\prod_{n\geq 1}\left(\frac{1-q^{pn}}{1-q^n}\right)^{24/(p-1)}, \qquad \psi(\tau) = p^{\frac{12}{p-1}}\varphi(\tau/p)$$

We have from [19] (see also [17, Sections 5,6])

$$\psi^p + \sum_{j=1}^{p}(-1)^j r_j \psi^{p-j} = 0, \tag{15}$$

where

$$(-1)^{j+1} r_j = p^{\frac{12}{p-1}+2}\sum_{m=j}^{p} b_m \varphi^{m-j+1} \tag{16}$$

with the following integer coefficients b_m

(17)

m	1	2	3	4	5	6	7
b_m for $p=5$	63	$52\cdot 5^3$	$63\cdot 5^5$	$6\cdot 5^8$	5^{10}		
b_m for $p=7$	82	$176\cdot 7^2$	$845\cdot 7^3$	$272\cdot 7^5$	$46\cdot 7^7$	$4\cdot 7^9$	7^{10}

It follows that, for $1 \leq j \leq p$,

$$r_j \in p^{\alpha_j} p\varphi\mathbb{Z}[p\varphi] \tag{18}$$

with

$$\alpha_j \geq j\frac{13-p}{p-1} + 2. \tag{19}$$

Let $r_j = 0$ for $j > p$. For $l \geq 1$ let

$$s_l = p^{\frac{12l}{p-1}+1} U(\varphi^l) = \sum_{j=0}^{p-1} \psi^l(\tau + j) \tag{20}$$

The coefficients r_j of (15) are invariant with respect to the variable change $\tau \mapsto \tau + 1$. It follows from the Newton-Girard formula that for $l \geq 1$

$$s_l = \sum_{j=1}^{l} (-1)^{j+1} r_j s_{l-j} \tag{21}$$

with the convention $s_0 = k$. In particular, $s_1 = r_1$, and it follows from (16) and (17) that

$$s_1 \in p^{\frac{12}{p-1}+1}(p\varphi)\mathbb{Z}[p\varphi]$$

Also, in the view of (20) and (16)

$$p^{-1}U(\varphi) - b_1\varphi \in (p\varphi)^2\mathbb{Z}[p\varphi]. \tag{22}$$

An induction argument which makes use of (21), (18), and (19) implies that

$$s_l \in p^{\frac{12l}{p-1}-l+2} p\varphi\mathbb{Z}[p\varphi],$$

and, in the view of (20),

$$U(\varphi^l) \in p^{1-l} p\varphi\mathbb{Z}[p\varphi].$$

We conclude that for a polynomial $P = \sum_{l=1}^{M} a_l x^{l-1} \in \mathbb{Z}[x]$

$$U(p\varphi P(p\varphi)) = U\left(\sum_{l=1}^{M} a_l p^l \varphi^l\right) = \sum_{l=1}^{M} a_l p^l U(\varphi^l) \in p^2\varphi\mathbb{Z}[p\varphi]. \tag{23}$$

The fact that the genus of $\Gamma_0(p)$ is zero implies (see [17, equation (2.7)] for details) the existence of integers $B_1, \ldots, B_{p^2}$ such that

$$U(j) = 744 + p^{\frac{13-p}{p-1}} \sum_{l=1}^{p^2} B_l p^{\frac{12(l-1)}{p-1}} \varphi^l = 744 + p^{\frac{13-p}{p-1}} (B\varphi + p\varphi P(p\varphi))$$

with $B = B_1 \in \mathbb{Z}$ and a polynomial $P(x) \in \mathbb{Z}[x]$. Equating the coefficients of q we find that $B \not\equiv 0 \bmod p$.

It follows that

$$p^{-\frac{13-p}{p-1}}(U(j - 744)) - B\varphi \in p\varphi\mathbb{Z}[p\varphi].$$

We now use (22), (23), and induction in a to conclude that

$$p^{-\frac{13-p}{p-1}-a}(U^a(j - 744)) - Bb_1^{a-1}\varphi \in p\varphi\mathbb{Z}[p\varphi],$$

which implies Proposition 2 since $b_1 \not\equiv 0 \bmod p$. □

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