HONDA - KANEKO CONGRUENCES AND THE MAZUR - TATE
\( p \)-ADIC \( \sigma \)-FUNCTION

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Abstract. Several years ago, Honda and Kaneko experimentally discovered remarkable congruences for Fourier coefficients of a certain weakly holomorphic modular form. The congruences attracted interest, and have been proved since then using various approaches. In this paper, we suggest yet another approach to these congruences. Specifically, the very existence of the \( p \)-adic \( \sigma \)-function constructed by Mazur and Tate allows us to prove and expand the congruences.

1. Introduction

Let \( q = \exp(2\pi i \tau) \) with \( \Im(\tau) > 0 \). Consider the Eisenstein series

\[
E_4(\tau) = 1 + 240 \sum_{n > 0} \left( \sum_{d \mid n} d^3 \right) q^n, \\
E_6(\tau) = 1 - 504 \sum_{n > 0} \left( \sum_{d \mid n} d^5 \right) q^n
\]

of weights 4 and 6 correspondingly. Let

\[
\eta(\tau) = q^{\frac{1}{24}} \prod_{n > 0} (1 - q^n)
\]

be Dedekind’s eta-function.

For a prime \( p \), denote by \( U_p \) Atkin’s \( U \)-operator which acts on formal power series by

\[
\left( \sum u(n)q^n \right) \Big| U_p = \sum u(pm)q^n.
\]

We say that a function \( \phi \) with a Fourier expansion \( \phi = \sum u(n)q^n \) with \( p \)-integral coefficients \( u(n) \in \mathbb{Q} \cap \mathbb{Z}_p \) is congruent to zero modulo a power of a prime \( p \),

\[
\phi = \sum u(n)q^n \equiv 0 \mod p^w,
\]

if all its Fourier expansion coefficients are divisible by this power of the prime, namely \( u(n) \equiv 0 \mod p^w \) for all \( n \).

The congruences

\[
\left( \frac{E_4(6\tau)}{\eta(6\tau)^4} \right) \Big| U_p^l \equiv 0 \mod p^l
\]

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for primes $p \equiv 1 \mod 3$ and integers $l \geq 0$ were discovered experimentally and conjectured by Honda and Kaneko in [7]. These congruences were considered in the context of weak harmonic Maass forms by the author in [5] and later fully proved, along with appropriate congruences for other primes, by Ahlgren and Andersen in [1] in the context of Hecke grids.

In this paper, we show how congruences (1) follow immediately from the existence of the $p$-adic $\sigma$-function introduced by Mazur and Tate in [16] (see also [4] for a recent alternative construction.) Furthermore, congruences (1) along with the absence of a constant term in the $q$-expansion

$$\frac{E_4(6\tau)}{\eta(6\tau)^4} = q^{-1} + 244q^5 + 3134q^{11} + 18760q^{17} + 84345q^{23} + 306252q^{29} + 98374q^{35} + \ldots$$

prompt us to consider an antiderivative (with respect to $D = \frac{1}{q} \frac{d}{dq}$) which we will multiply by $\eta(6\tau)^4$. Specifically we define

$$F := \eta(6\tau)^4 \int \frac{E_4(6\tau)}{\eta(6\tau)^4} \frac{dq}{q},$$

where the antiderivative $\int \frac{E_4(6\tau)}{\eta(6\tau)^4} dq/q$ is taken term-wise for the $q$-expansion and has a zero constant term.

Remark 1. Note that $F \in (\mathbb{Q} \cap \mathbb{Z}_p)[[q]]$ for every prime $p \equiv 1 \mod 3$ is equivalent to congruences (1) because

$$\left(\frac{E_4(6\tau)}{\eta(6\tau)^4}\right) = q \frac{d}{dq} \left(\frac{F}{\eta(6\tau)^4}\right).$$

The first assertion of the theorem below is thus equivalent to (1) whereas the second one is independent and illustrates the distinction between our approach and those used in the literature.

**Theorem 1.** Let $p \equiv 1 \mod 3$ be a prime. Then we have

$$F \in (\mathbb{Q} \cap \mathbb{Z}_p)[[q]].$$

Furthermore, for all integers $m \geq 0$ we have

$$F|U_p^m \equiv F|U_p^{m+1} \mod p^{m+1}.$$  

Remark 2. Our proof of Theorem 1 indicates that there is nothing important about the choice of $E_4(6\tau) \in M_4(\Gamma_0(36))$ besides this choice looks very nice and is probably meaningful in the context of the work of Kaneko and Koike [8, 9, 10, 11]. One may substitute for $E_4(6\tau)$ any element of a $\mathbb{Z}$-submodule in $M_4(\Gamma_0(36)) \cap \mathbb{Z}[\eta]$ of rank at least four of forms with integral $q$-expansion coefficients so that Theorem 1 remains valid.

Remark 3. The function $F$ defined in (2) as a function on the upper half-plane is a mixed mock modular form (see [3, Section 7.3] for the definition). This function is closely related to a mixed mock modular solution of the Kaneko - Zagier equation found in [6]. We do not investigate the relation in this paper.

In this paper we primarily stick to the example related to the elliptic curve $X_0(36)$ (with the corresponding weight 2 eigenform $g = \eta(6\tau)^4$) which is of special interest in view of the work of Kaneko and Koike cited above.
Analogs of Theorem 1 can be obtained using our approach in the cases of $X_0(32)$ (with $g = \eta(4\tau)^2\eta(8\tau)^2$) and $X_0(27)$ (with $g = \eta(3\tau)^2\eta(9\tau)^2$). Besides $X_0(36)$, we consider only $X_0(49)$ in this paper. Theorem 2 below illustrates independence of our argument on the $\eta$-product representation of a weight 2 modular form.

**Theorem 2.** Let

$$h = q + q^2 - q^4 - 3q^8 - 3q^9 + 4q^{11} - q^{16} - 3q^{18} + 4q^{22} + 8q^{23} - 5q^{25} + 2q^{29} + 5q^{32} + \ldots \in S_2(\Gamma_0(49))$$

be the unique normalized eigenform of weight 2 and level 49.

There exists $H \in M_4(\Gamma_0(49))$ with integral Fourier coefficients and constant term 1 which has the following properties.

The $q$-expansion of $H/h$ has no constant term so that a term-wise antiderivative $\int (H/h) dq/q$ makes sense. We choose the antiderivative to have the constant term of $1/2$ and let

$$G := h \int \frac{H dq}{hq}.$$ 

Then

$$G \in (Q \cap Z_p)[[q]]$$

and

$$G|U_p^m \equiv G|U_p^{m+1} \mod p^{m+1}$$

for every $p$ which splits in $Q(\sqrt{-7})$ and integers $m \geq 0$.

**Remark 4.** As in Remark 1, the condition $G \in (Q \cap Z_p)[[q]]$ is equivalent to the analog of congruences (1)

$$\left(\frac{H}{h}\right)|U_p^l \equiv 0 \mod p^l$$

for every $p$ which splits in $Q(\sqrt{-7})$ and integers $l \geq 0$.

**Remark 5.** A modular form $H \in M_4(\Gamma_0(49))$ whose existence is claimed in the Theorem 2 is explicitly constructed in Section 3 below in the proof of Theorem 2 and has a Fourier expansion

$$H = 1 + q - q^3 + q^4 + q^5 + q^6 - 5q^7 - 5q^8 - q^9 + 6q^{10} - 7q^{11} - 7q^{12} - 7q^{13} + 4q^{14} + 2q^{15} + 8q^{16} - 8q^{17} + 16q^{18} + 15q^{19} + 14q^{20} + 7q^{21} + \ldots$$

Contrary to the case of Theorem 1, this modular form does not belong to the Eisenstein series subspace of $M_4(\Gamma_0(49))$. (That may be checked numerically using e.g. magma which allows one to easily construct a basis of the Eisenstein series subspace). Similarly to the situation in Theorem 1, the modular form $H$ such that the statement of Theorem 2 holds true is not unique. The author does not know whether it is possible to choose $H$ in the Eisenstein series subspace.

The paper is organized as follows. In Section 2, we introduce necessary notations, quote a specialization to our case of a proposition proved by Mazur and Tate in [16], and derive a couple of corollaries from this proposition for further applications. We apply these corollaries to prove Theorems 1 and 2 in Section 3.
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2. The $p$-adic $\sigma$-function

In this Section, we briefly recall and specialize to our setting some facts about the $p$-adic $\sigma$-function discovered by Mazur and Tate in [16]. We refer to the original paper [16] for details.

Let

$$g = \sum_{n \geq 1} b(n)q^n \in S_2(\Gamma_0(N)) \cap q(1 + \mathbb{Z}[[q]])$$

be a weight two normalized cusp Hecke eigenform with integral coefficients (we will later specialize $g = \eta(6\tau)^4$ or $g = h$ for Theorems 1 or 2 respectively). The Eichler - Shimura congruence relation allows us to consider an elliptic curve $E$ defined over $\mathbb{Q} \subset \mathbb{Q}_p$ with a non-vanishing differential

$$\omega = g \frac{dq}{q}.$$ 

Assume that $p > 2$ is an ordinary prime (i.e. $b(p) \neq 0$). Then the formal group $E^f$ of $E$ defined by its logarithm

$$E^f = \sum_{n \geq 1} b(n)\frac{q^n}{n}$$

is isomorphic to the formal multiplicative group. Here we choose $q$ to be the parameter on $E^f$. Since $E$ is defined over $\mathbb{Q}$, we have the Weierstrass $\wp$-function

$$\wp(E) = q^{-2} + \ldots \in \mathbb{Q}((q)).$$

The canonical eks-function $X$ differs from the Weierstrass $\wp$-function on $E$ by the value of of the $p$-adic weight 2 Eisenstein series $P/12$ at $(E, \omega)$:

$$X - \wp = -\frac{1}{12}P(E, \omega).$$

(In this formula, the negative sign on the right side is missing in [16, p.665]. See also [15, footnote, p.581].)

In the presence of complex multiplication, Katz’s $p$-adic analogue of Damerell’s theorem ([13, Section 8.0]) implies that this quantity is algebraic, not merely $p$-adic, and therefore the same value can be taken for every prime $p$ as soon as the prime is ordinary. Specifically, let

$$E_2(\tau) = 1 - 24\sum_{n>0} \left( \sum_{d|n} d \right) q^n \quad \text{and} \quad E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi y},$$

with

$$E_2(\tau) = 1 - 24\sum_{n>0} \left( \sum_{d|n} d \right) q^n \quad \text{and} \quad E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi y}.$$
where \( y = \Im(\tau) \). Then it follows from \([13, \text{Comparison Theorem 8.0.9}]\) that
\[
P(E, \omega) = \begin{cases} 
\frac{-4\pi^2}{12} E_2^* \left( (1 + i\sqrt{3})/2 \right) = 0 & \text{if } g = \eta(6\tau)^4 \text{ (and } N = 36) \\
\frac{-4\pi^2}{12} E_2^* \left( (1 + i\sqrt{7})/2 \right) = -3 & \text{if } g = h \text{ (and } N = 49),
\end{cases}
\]
where \( \Omega \) is the real period of \( E \). We thus have that
\[
X = q^{-2} + \ldots \in \mathbb{Q}(\!(q)\!).
\]

In this setting, the construction of Mazur and Tate produces the \( p \)-adic \( \sigma \)-function \( \sigma \in \mathbb{Z}_p[[q]] \) which has the following property (see Proposition 2 in \([16, \text{Section 4.2}]\)):
\[
(5) 
X = \frac{d}{\omega} \left( -\frac{1}{\sigma/\omega} \right) = \frac{d}{d\mathcal{E}} \left( -\frac{1}{\sigma/\mathcal{E}} \right) = -\frac{q}{g} \frac{d}{dq} \left( \frac{q d \log \sigma}{g dq} \right).
\]

We invert the formal power series in (3) to obtain \( q \in \mathcal{E}(1 + \mathbb{Q}[[\mathcal{E}]]) \), and plug it in for \( q \) into \( \sigma \in \mathbb{Z}_p[[q]] \) to obtain that
\[
\sigma(\mathcal{E}) = \mathcal{E} + o(\mathcal{E}) \in \mathcal{E}(1 + \mathbb{Q}_p[[\mathcal{E}^2]])
\]
because, by \([16, \text{Theorem 3.1(I)}]\), the function \( \sigma \) is odd and satisfies \( d\sigma/\omega \mid_0 = 1 \). We plug in back the expression (3) for \( \mathcal{E} \) to derive that
\[
\sigma \in q(1 + \mathbb{Q}_p[[q]]) \cap \mathbb{Z}_p[[q]] = q(1 + \mathbb{Z}_p[[q]]).
\]

Since \( X = \varphi - P(E, \omega)/12 \in \mathcal{E}^{-2} + \mathbb{Q}[[\mathcal{E}]] \) and \( X = \frac{d}{d\mathcal{E}} \left( -\frac{1}{\sigma/\mathcal{E}} \right) \), we find that \( \sigma = \exp(\log(\sigma)) \in \mathbb{Q}[[\mathcal{E}]] \). We now again plug in the expression (3) for \( \mathcal{E} \) and find that \( \sigma \in \mathbb{Q}[[q]] \). We conclude that under the conditions of our Theorems 1 and 2 the \( p \)-adic \( \sigma \)-function actually belongs to \( \mathbb{Q}[[q]] \cap q(1 + \mathbb{Z}_p[[q]]) = q(1 + \mathbb{Q}_p \cap \mathbb{Z}_p)[[q]] \).

We now summarize the above remarks in a proposition for later use.

**Proposition 1.** Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \), and let \( \omega = q^{-1}gdq \), where \( g \) is the associated weight 2 normalized Hecke eigenform. Assume that \( P(E, \omega) \in \mathbb{Q} \).

There exists \( \sigma \in q(1 + \mathbb{Q}[[q]]) \) such that, for every ordinary prime \( p > 2 \), the formal power series \( \sigma \) coincides under the embedding \( \mathbb{Q} \hookrightarrow \mathbb{Q}_p \) with the \( p \)-adic \( \sigma \)-function from \([16]\) associated with \( (E, \omega) \).

In particular, for every such prime \( p \)
\[
\sigma \in q(1 + (\mathbb{Q} \cap \mathbb{Z}_p)[[q]]),
\]
and
\[
\frac{q}{g} \frac{d}{dq} \left( \frac{q d \log \sigma}{g dq} \right) = -X,
\]
where \( X = \varphi - \frac{1}{12} P(E, \omega) \in (\mathbb{Q} \cap \mathbb{Z}_p)((q)) \) is the canonical eks-function from \([16]\).

We now derive two corollaries from Proposition 1. The first one is immediate.

**Proposition 2.** Under the assumptions of Proposition 1, we have that \( Xg \in (\mathbb{Q} \cap \mathbb{Z}_p)((q)) \) and
\[
(Xg)|U_p^l \equiv 0 \mod p^l
\]
for every ordinary prime \( p > 2 \) and every integer \( l \geq 0 \).
Proof. By Proposition 1, we have that $\sigma \in q(1 + \mathbb{Z}_p[[q]])$. Since also $g \in q(1 + \mathbb{Z}[[q]]) \subset q(1 + \mathbb{Z}_p[[q]])$,

$$\left( g \frac{d \log \sigma}{dq} \right) = \frac{q}{g} \frac{d \sigma}{dq} \in \mathbb{Z}_p[[q]],$$

and the claim follows immediately from

$$Xg = -q \frac{d}{dq} \left( g \frac{d \log \sigma}{dq} \right).$$

\[\square\]

For the second corollary, let

$$\zeta^* = \frac{d \log \sigma}{d\mathcal{E}} = \frac{q \frac{d \log \sigma}{g \frac{dk}{dq}}}{dq}.$$  

As a formal power series in $q$, our $\zeta^*$ is the Weierstrass $\zeta$-function $\zeta(\mathcal{E}(q))$ plus a multiple of $\mathcal{E}$ chosen in a way such that $\zeta^* \in \mathbb{Z}_p((q))$. In particular, as power series in $\mathcal{E}$,

$$\frac{d\zeta^*}{d\mathcal{E}} = -X, \text{ and } \zeta^* = -\int X \, d\mathcal{E},$$

where the formal term-wise antiderivative assumes the constant term of the series to be zero (so that both $\zeta$ and $\zeta^*$ are odd formal power series in $\mathcal{E}$). Note that, while $\zeta^*$ has no constant term as a power series in $\mathcal{E}$, it may have a non-zero constant term, which is the constant term of $1/\mathcal{E}(q)$ as a power series in $q$. That is why the constant term of $1/2$ appears in Theorem 2.

**Proposition 3.** Under the assumptions of Proposition 1, let

$$G = g\zeta^*.$$  

Then

$$G \in (\mathbb{Q} \cap \mathbb{Z}_p)[[q]]$$

and

$$G|_{U^m_p} \equiv G|_{U^{m+1}_p} \mod p^{m+1}$$

for every ordinary prime $p > 2$ and every integer $m \geq 0$.

Proof. While $G \in (\mathbb{Q} \cap \mathbb{Z}_p)[[q]]$ is immediate from Proposition 1 and the definitions of $\zeta^*$ and $g$, the congruences require a proof.

Let

$$q \frac{d \log \sigma}{dq} = g\zeta^* = G = 1 - \sum_{n \geq 1} c(n)q^n.$$  

The congruences we want to prove are

$$\frac{1}{n} (c(n) - c(n/p)) \in \mathbb{Z}_p$$

for any integer $n \geq 1$ with the usual convention $c(n/p) = 0$ if $p \nmid n$. Since $\sigma \in q(1 + \mathbb{Z}_p[[q]])$, by a standard induction argument, it can be written as an infinite product

$$\sigma = q \prod_{n \geq 1} (1 - q^n)^{\alpha_n} \text{ with } \alpha_n \in \mathbb{Z}_p.$$
We thus have that

\[ G = q \frac{d \log \sigma}{dq} = 1 - \sum_{n \geq 1} \alpha_n \frac{n q^n}{1 - q^n} = 1 - \sum_{k \geq 1} \left( \sum_{n | k} n \alpha_n \right) q^k, \]

and equating like powers of \( q \) we obtain

\[ c(n) = \sum_{d | n} d \alpha_d. \]

The Möbius inversion formula now implies that

\[ \alpha_n = \frac{1}{n} \sum_{d | n} \mu(d) c(n/d). \]

In particular, for \( n = p^u \) we obtain

\[ p^{-u} \left( c(p^u) - c(p^{u-1}) \right) = \alpha_{p^u} \in \mathbb{Z}_p. \]

Now write \( n = p^u m \) with \( p \nmid m \), and an induction argument in the number of prime divisors of \( m \) yields (6).

\[ \square \]

**Remark 6.** Our elementary argument above can be easily inverted. Namely, the congruences (6) imply \( \sigma \in q(1 + \mathbb{Z}_p[[q]]) \). Interestingly, there is an alternative way to prove that which we now sketch. Dwork’s lemma (see e.g. [14, Lemma IV.3]) states that \( \Phi \in 1 + X \mathbb{Q}_p[[X]] \) is actually in \( 1 + X \mathbb{Z}_p[[X]] \) if and only if \( \Phi(X^p)/\Phi(X)^p \in 1 + pX \mathbb{Z}_p[[X]] \). We notice that

\[ \sigma = \exp \left( \int \frac{1}{q} G \ dq \right) \]

and apply Dwork’s lemma to \( \sigma \) to conclude that \( \sigma \in q(1 + \mathbb{Z}_p[[q]]) \) is equivalent to

\[ \exp(\phi) = 1 + \sum_{m \geq 1} \frac{\phi^m}{m!} \in \mathbb{Z}_p[[q]] \]

with

\[ \phi = p \sum_{n \geq 1} \left( \frac{c(n) - c(n/p)}{n} \right) q^n. \]

One of the fundamental lemmas in the theory of crystalline cohomology [12, Key Lemma 5.1.3] allows one to derive (7) from

\[ \phi \in pq \mathbb{Z}_p[[q]], \]

and the latter condition coincides with (6).
3. Proofs of Theorems 1 and 2

Let \( E \) be the elliptic curve (defined over the rationals) which is isomorphic to a modular curve \( E \cong \Gamma_0(N) \backslash \mathfrak{H} \), where \( N = 36 \) in Theorem 1 and \( N = 49 \) in Theorem 2. We recall a description of this isomorphism (see e.g. [17, Section 3] for details). Let

\[
g(\tau) = \sum_{n \geq 1} b(n) q^n \in S_2(N), \quad \text{with} \quad q = \exp(2\pi i \tau) \quad \text{and} \quad \tau \in \mathfrak{H}, \text{the complex upper half-plane},
\]

be the weight two normalized (i.e. \( b(1) = 1 \)) cusp Hecke eigenform associated with \( E \) (that is \( g = \eta(6\tau)^4 \) or \( g = h \) for Theorems 1 or 2 respectively), and let

\[
\mathcal{E}(\tau) = \sum_{n \geq 1} \frac{b(n)}{n} q^n
\]

be the Eichler integral associated with \( g \). The isomorphism in question is the map

\[
\Gamma_0(N) \backslash \mathfrak{H} \quad \tau \mapsto \quad (\varphi(\mathcal{E}(\tau)), \varphi'(\mathcal{E}(\tau))),
\]

where \( \varphi(z) \) is the Weierstrass \( \varphi \)-function (and \( \varphi'(z) = \frac{d}{dz} \varphi(z) \) is its derivative), and the equation of \( E \) in Weierstrass form is

\[
\varphi'^2 = 4\varphi^3 - g_2 \varphi - g_3.
\]

While in general the described map is merely a finite covering, it has degree 1 (see, e.g. [2]), and is therefore an isomorphism when \( N = 36 \) and \( 49 \). That means, in particular, that the Eichler integral \( \mathcal{E}(\tau) \) (as a function of \( \tau \)) in these cases takes the value of zero at infinity only (neither in other cusps nor in the interior of \( \mathfrak{H} \)). It follows that, in both cases, the function \( \varphi(\mathcal{E}(\tau)) \) on \( \mathfrak{H} \) is modular (of weight zero) with its only pole (of order 2) at infinity, and therefore we have weight 4 holomorphic modular forms

\[
g(\tau) \varphi(\mathcal{E}(\tau)) \in M_4(\Gamma_0(N)),
\]

and it follows from (4) that the \( q \)-series

\[
H := g^2(\varphi(\mathcal{E})) - P(E, \omega)/12 = g^2 X \in M_4(\Gamma_0(N)),
\]

where

\[
P(E, \omega)/12 = \begin{cases} 
0 & \text{if } N = 36 \\
-1/4 & \text{if } N = 49.
\end{cases}
\]

Under the assumptions of Theorem 2, ordinary primes are exactly those which split in \( \mathbb{Q}(\sqrt{-7}) \).

Now Propositions 2 and 3 imply all assertions of Theorem 2 except \( H \in \mathbb{Z}[[q]] \) (we merely have \( H \in (\mathbb{Q} \cap \mathbb{Z}_p)[[q]] \) for all splitting primes \( p > 2 \)).

Numerically, for \( N = 49 \) and \( g = h \), it is easy to find

\[
H = 1 + q - q^3 + q^4 + q^5 + q^6 - 5q^7 - 5q^8 - q^9 + 6q^{10} - 7q^{11} - 7q^{12} - 7q^{13} + 4q^{14} + 2q^{15} + 8q^{16} + \ldots,
\]

and check (using e.g. magma) that indeed \( H \in M_4(\Gamma_0(49)) \cap \mathbb{Z}[[q]] \). Namely, it is already proved that \( H \in M_4(\Gamma_0(49)) \), and magma provides a convenient basis of this space (note that \( \dim M_4(\Gamma_0(49)) = 18 \)) which consists of modular forms with integral Fourier coefficients.
It is thus easy to present \( H \) as an integral linear combination of these forms proving that 
\( H \in \mathbb{Z}[[q]]. \)

The following numerics illustrate Theorem 2. Theorem 2 predicts congruences for

\[
H/h = q^{-1} + q^{3} + q^{5} - 2q^{6} - 4q^{10} - 4q^{12} - 3q^{13} + 4q^{17} + 2q^{19} + 4q^{20} + 10q^{26} + 6q^{27} + 2q^{31} - 11q^{33} + \ldots ,
\]

and it is easy to observe these congruences for e.g. \( p = 11 \). We now choose the antiderivative \(\int (H/h) dq/q = -q^{-1} + \frac{1}{2} + \frac{1}{3} q^{3} + \ldots\) and obtain \(G = -h \int (H/h) dq/q:\n\]
\[
G = 1 + \frac{1}{2} q - \frac{1}{2} q^{2} - q^{3} + \frac{1}{6} q^{4} - \frac{1}{3} q^{5} - \frac{1}{5} q^{6} - \frac{38}{15} q^{7} - \frac{7}{6} q^{8} + \frac{17}{10} q^{9} + \frac{11}{3} q^{10} - \frac{3}{5} q^{11} + \frac{7}{5} q^{12} + \ldots .
\]

For \( p = 11 \),
\[
G|U_{11} = 1 - \frac{3}{5} q + \frac{931}{195} q^{2} + \frac{8882}{2635} q^{3} + \frac{8003}{1235} q^{4} - \frac{212411}{104481} q^{5} - \frac{1959744}{1404455} q^{6} + \frac{645535522841084}{6309690542235} q^{7} + \ldots
\]
illustrating the second congruences \( G - G|U_{11} \equiv 0 \pmod{11} \) predicted by Theorem 2.

We now turn to the proof of Theorem 1. To this end, with \( N = 36 \) and \( g = \eta(6\tau)^{4} \) we have
\[
H = q^{2} \varphi(E) = g^{2} X \in M_{4}(\Gamma_{0}(36))
\]
(recall that now \( P(E, \omega) = 0 \)). Splitting primes \( p \) for \( E = X_{0}(36) \) are exactly the primes \( p \equiv 1 \pmod{3} \). Numerically,
\[
H = 1 - 7q^{6} + 13q^{12} + 11q^{18} - 43q^{24} - 18q^{30} + 31q^{36} + 184q^{42} - 59q^{48} - 367q^{54} - 90q^{60} + 180q^{66} + \ldots,
\]

and Proposition 2 implies
\[
(8) \quad \left( \frac{H}{g} \right)|_{U_{p}^{l}} \equiv 0 \pmod{p^{l}}
\]
for every prime \( p \equiv 1 \pmod{3} \) and \( l \geq 0 \). Furthermore, the subtlety with the choice of antiderivative is invisible in this case because \( g = \eta(6\tau)^{4} \) has a zero coefficient of \( q^{2} \), and simply \(\int (H/h) dq/q = -q^{-1} - (3/5)q^{5} - (1/11)q^{11} + \ldots\) for
\[
H/g = q^{-1} - 3q^{5} - q^{11} + 5q^{17} + 8q^{23} + q^{29} - 28q^{35} - 11q^{41} + 10q^{47} + 41q^{53} + 41q^{59} - 26q^{65} - 53q^{71} + \ldots .
\]

We also have \( G = -g \int (H/g) dq/q:\n\]
\[
G = 1 - \frac{17}{5} q^{6} - \frac{17}{55} q^{12} + \frac{7987}{935} q^{18} + \frac{17429}{21505} q^{24} - \frac{686526}{124729} q^{30} - \frac{9332977}{623645} q^{36} - \frac{68432264}{25569445} q^{42} + \ldots
\]
and Proposition 3 implies
\[
(9) \quad G|U_{p}^{m} \equiv G|U_{p}^{m+1} \pmod{p^{m+1}}
\]
for every prime \( p \equiv 1 \pmod{3} \) and \( m \geq 0 \). This is, however, not what we wanted since \( H \neq E_{4}(6\tau) \) and therefore \( G \neq F \), where
\[
F = -1 + \frac{264}{5} q^{6} + \frac{4824}{55} q^{12} + \frac{50016}{935} q^{18} + \frac{4693992}{21505} q^{24} + \frac{17218800}{124729} q^{30} + \frac{254186784}{623645} q^{36} + \ldots
\]
is defined by (2). A drastic simplification
\[
12G + F = 11 + 12q^{6} + 84q^{12} + 156q^{18} + 228q^{24} + 72q^{30} + 228q^{36} + 96q^{42} + \ldots.
\]
was observed in [5] and follows from the identity of functions in $\tau$

\begin{equation}
12G + F = E_2(6\tau) - 2E_2(12\tau) + \frac{3}{2}(E_2(6\tau) - 3E_2(18\tau)) - 3(E_2(6\tau) - 6E_2(36\tau)).
\end{equation}

Note that, although neither $G$ nor $F$ are modular, the right hand side belongs to $M_2(36)$.

We have that $G \in (\mathbb{Q} \cap \mathbb{Z}_p)[[q]]$ for every prime $p \equiv 1 \mod 3$ by Proposition 3. This fact along with the obvious observation that the right-hand side of (10) has all its $q$-expansion coefficients in $\mathbb{Z}_p$ for primes $p > 2$ implies $F \in (\mathbb{Q} \cap \mathbb{Z}_p)[[q]]$ (and also (1) by Remark 1).

Since for $p \nmid M$ obviously

$$E_2(M\tau)|U^m \equiv E_2(M\tau)|U^{m+1} \mod p^{m+1},$$

equations (9) and (10) imply the congruences claimed in Theorem 1.

\[\blacksquare\]

\textbf{References}


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