A note on quadratic congruences.

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The notion of quadratic congruences was introduced recently in [1]. In the present note we revise the original definition and present an explanation of the the phenomena.

The Fourier coefficients of half-integral weight modular forms is a very interesting object to study. Even in the simplest case of Eisenstein series these numbers bare non-trivial information. Identities and congruences for them can mirror various phenomena. The notion of quadratic congruences was introduced in recently appeared paper [1]. Several examples were presented in loc. cit. . In the present note we revise the original definition and explain the phenomena. We prove the existence of infinite families of modular forms of half-integral weight which satisfy the congruences. In particular, for a cusp Hecke eigenform of half-integral weight in Kohnen's "+"-space we indicate a finite list of primes. The quadratic congruences might appear modulo these primes. After that, the congruences could be checked by machine computation.

Let $\varphi = \sum_{n\geq 0} c(n)q^n$ be a modular form on $\Gamma_0(4)$ of half-integral weight $k+1/2$. We assume φ to be a Hecke eigenform in the Kohnen's subspace $M^+_{k+1/2}$ [7]. This means that $c(n) = 0$ unless $(-1)^k n \equiv 0$ or $1 \bmod 4$, and for a fundamental discriminant D (i.e. 1 or a discriminant of quadratic field) with $(-1)^k D > 0$,

$$c(n^2|D|) = c(|D|)\sum_{d|n}\mu(d)\chi_D(d)d^{k-1}a(n/d). \tag{1}$$

Here $a(n)$ are the Hecke eigenvalues (i.e. the Fourier coefficients of the corresponding normalized cusp form of even weight), μ is the Möbius function, χ_D is the quadratic character associated with $\mathbb{Q}(\sqrt{D})$. We can and will assume the Fourier coefficients $c(n)$ to be algebraic numbers.

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers. Fix once and for all an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ of the field of algebraic numbers into the Tate field (the completion of an algebraic closure $\overline{\mathbb{Q}_p}$). We will tacitly identify algebraic numbers with their images under ι. We denote by $v_p(\cdot)$ the p-adic ordinal.

Definition.

Let p be an odd prime. Let γ be 1 or -1. For a positive integer d denote by ξ_d the quadratic Dirichlet character associated with $\mathbb{Q}(\sqrt{-d})$.

a. *We say that φ satisfies type a quadratic congruences modulo p if* $k \equiv 1 \bmod (p-1)p^{r-1}$

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for an integer $r \geq 1$, and

$$\xi_d(p) = \gamma \quad \textit{yields} \quad v_p(c(d)) \geq 1.$$

b. *We say that φ satisfies type b quadratic congruences modulo p if $k \equiv 1 \bmod (p-1)p^{r-1}/2$ for an integer $r \geq 1$, $k \not\equiv 1 \bmod p-1$, and*

$$\xi_d(p) = \gamma \quad \textit{yields} \quad v_p(c(pd)) \geq 1.$$

Remarks.

Our definition is slightly different from the original one given in [1]. However, it covers the examples presented in loc. cit. .

Notice that (1) allows to check the congruences only when $-d$ in case a (or, respectively, $(-1)^k pd$ in case b) is a fundamental discriminant.

Let $\mathcal{H}_k = \sum H(k,N)q^N$ be the Cohen - Eisenstein series [3]. This is the unique Eisenstein series in $M^+_{k+1/2}$. The following assertion might also be considered as motivating for our definition.

Proposition 1 *The Cohen - Eisenstein series $\mathcal{H}_k$ satisfy the quadratic congruences (of both a and b types) modulo p as soon as $k \equiv 1 \bmod (p-1)/2$.*

One can find several specific examples of this phenomena in [1].

The proof uses just the definition of Cohen numbers $H(k,N)$ [3], and standard interpretation of the Kubota - Leopoldt p-adic zeta-function [11], section 3.3. We omit this proof.

From now on we assume that φ is a cusp form. Note that the (standard) bounded denominators argument allows to normalize φ so that its Fourier coefficients are p-integral. We assume this normalization. Let us state the main result of the note.

Theorem. *Let φ be as above.*

Assume that the space $S_2(\Gamma_0(p))$ of cusp forms of weight 2 on $\Gamma_0(p)$ is not empty, and there are no modulo p congruences between normalized (i.e. $a(1) = 1$) eigenforms in this space.

Assume that the Hecke eigenvalue $a(p)$ is a p-adic unit.

Then φ satisfies quadratic congruences modulo p (of both a and b types) as soon as $k \equiv 1 \bmod (p-1)p^{r-1}/2$ and r is sufficiently large.

Proof. Since we have supposed that $S_2(\Gamma_0(p))$ is not empty, we can restrict ourselves with $p \geq 11$. Fix the topological generator $u = 1+p$ of $\Gamma = 1 + \mathbb{Z}_p$. Let K be a finite extension of $\mathbb{Q}_p$, and $\mathcal{O}_K$ its p-adic integer ring. Put $\Lambda = \mathcal{O}_K[[X]]$. We refer to [5], Chapter 7, for the basic facts and detailed explanations concerning Λ-adic forms. Denote by $f = \sum_{n\geq 1} a(n)q^n \in S_{2k}(SL_2(\mathbb{Z}))$ the normalized Hecke eigenform connected with φ via the Shimura correspondence [8]. Let α and β be the roots of its p-th Hecke polynomial: $\alpha + \beta = a(p)$, $\alpha\beta = p^{2k-1}$ and $v_p(\alpha) = 0$. We claim that there exists a p-ordinary Λ-adic normalized cusp eigenform $F(X) = \sum_{n\geq 1} A_n(X)q^n \in \mathcal{O}_K[[X]][[q]]$ with the following specializations:

$$F(u^{2k}-1) = f^* = f(z) - \beta f(pz),$$

$$F(u^2-1)=\phi.$$

Here $\phi=\sum_{n\geq 1}a_2(n)q^n$ is a normalized eigenform in $S_2(\Gamma_0(p))$.

Indeed, take sufficiently large finite field $\mathbb{F}$ of characteristic p, and write O for the ring of Witt vectors with coefficients in $\mathbb{F}$. Let $h_2=h_2(\Gamma_0(p),id;O)$ be the Hecke algebra defined as in [5], Section 5.3. Here id stands for the trivial character. Since we assumed that there are no modulo p congruences between eigenforms in $S_2(\Gamma_0(p))$, we have $h_2\cong O\times O\times\cdots\times O$ as O-algebras. Let h be the universal p-ordinary Hecke algebra ([5], Section 7.3) specializing to h_2, that is, $h/(X-u^2+1)\cong h_2$. Then by Hensel's lemma ([2], 3.4.6) each idempotent of h_2 can be lifted to h, and hence $h\cong\Lambda\times\Lambda\times\cdots\times\Lambda$ as Λ-algebras. This decomposition is compatible with the one at weight 2. The author indebted to Professor H. Hida for the argument above. This clarifies a remark in [5], p. 221. It follows now from [5], Theorem 7.3.6 (and its proof) that the space of cusp ordinary forms $S^{ord}(id;\Lambda)$ has basis consisting of normalized eigenforms. Our claim follows now from ([5], Theorem 7.3.3) since both f^* and ϕ are (complex-analytic) normalized eigenforms (these are p-stabilized newforms in the terminology of [4]). It also follows from [5], Theorem 7.3.3, that the specialization of F at any arithmetic point P is a normalized p-ordinary eigenform f_P. In other words, F gives rise to a p-adic analytic family of cusp forms. Here $P=(2k,\varepsilon)$ is a pair of an even integer and a character $\varepsilon:\Gamma\to\mathcal{O}_K$. Let χ be a Dirichlet character. The twisted L-series of a (complex-analytic) normalized cusp form $\psi=\sum_{n\geq 1}b(n)q^n$ is defined by the analytic continuation of $L(\psi,\chi,s)=\sum_{n\geq 1}\chi(n)b(n)n^{-s}$. Its special values are known to become algebraic after dividing by a common transcendental factor. The theorem on two-variable p-adic interpolation of these numbers in the setting appropriate for our purposes is proven in [6]. We quote the result.

Proposition 2 *[6], Theorem 1.*

Let Δ be a positive integer prime to p, and let $\mathbb{Z}_\Delta$ be the projective limit of $(\mathbb{Z}/p^m\Delta\mathbb{Z})^*$. Let F be a p-adic analytic family of normalized cusp eigenforms.

There exists a $\mathcal{O}_K$-valued measure μ on $\mathbb{Z}_\Delta\times\Gamma$ such that for each primitive Dirichlet character χ of conductor Δp^m with $m\geq 0$, an arithmetic point $P=(2k,\varepsilon)$ with $k\geq 1$ and $0\leq\nu\leq 2k-2$,

$$\int_{\mathbb{Z}_\Delta\times\Gamma}\chi^{-1}(z)z^\nu\varepsilon(w)w^{2k}d\mu(z,w)=$$

$$A_p(P)^{-m}(-\Delta)^\nu p^{\nu m}(1-A_p(P)^{-1}\chi^{-1}(p)p^\nu)\mathcal{E}_P^\pm\frac{G(\chi^-1)\nu!L(f_P,\chi,\nu+1)}{(2\pi i)^\nu\Omega_{f_P}^\pm}\qquad(2)$$

Here $\Omega_{f_P}^\pm\in\mathbb{C}^*$ is determined up to p-adic unit. $\mathcal{E}_P^\pm\in\mathcal{O}_K\backslash\{0\}$ is the error term of the p-adic interpolation. The function $P\to v_p(\mathcal{E}_P^\pm)$ is locally constant and of finite range. The sign "$\pm$" agrees with that of $\chi(-1)(-1)^\nu$. $G(\chi^{-1})$ denotes the Gauss sum of χ^{-1}.

Put $\nu=k-1$ in (2). Then the central critical values of L-function appear in the right. These are intimately connected with Fourier coefficients of modular forms of half-integral weight [8]. Namely, if φ and f as above are related to each other via the Shimura correspondence, one has

$$c(|D|)^2 = \frac{<\varphi,\varphi>}{<f,f>}\frac{(k-1)!}{\pi^k}|D|^{k-1/2}L(f,\chi_D,k),$$

where χ_D is the quadratic character associated with $\mathbb{Q}(\sqrt{D})$, and D is the discriminant of this quadratic field, $(-1)^k D > 0$. Rewrite the above formula as

$$c(|D|)^2 = \Lambda(f)^{-1}\frac{(k-1)!}{\pi^k}|D|^{k-1/2}L(f,\chi_D,k),$$

and assume the normalization constant $\Lambda(f)$ to be such that $\inf_D v_p(c(|D|)) = 0$.

For the positive integer d we put $\xi = \xi_d$ if $k \equiv 1 \bmod (p-1)p^{r-1}$ (case a). If $k \equiv 1 \bmod (p-1)p^{r-1}/2$ and $k \not\equiv 1 \bmod p-1$ (case b), we denote by ξ the quadratic Dirichlet character associated with $\mathbb{Q}(\sqrt{(-1)^k dp})$. Let Δ be the absolute value of the discriminant of $\mathbb{Q}(\sqrt{-d})$. Since $-d \equiv (-1)^k dp$, the character ξ is defined modulo Δp correctly. Put $\epsilon = 1$ if $2|k$ and $\epsilon = i$ if $2 \not| k$. This is the sign of the Gauss sum $G(\xi)$. The number

$$R(k) = (-2i)^{1-k}\epsilon\mathcal{E}_P^{\pm}\frac{\Lambda(f)}{\Omega_{f_P}^{\pm}}$$

is algebraic and depends only on our Λ-adic form F and arithmetic point $P = (2k, id)$. Let us evaluate the integral (2) at this point:

$$\int_{\mathbb{Z}_\Delta\times\Gamma}\xi(z)z^{k-1}w^{2k}d\mu(z,w) = \alpha^{-1}(1-\xi(p)\beta p^{-k})(1-\alpha^{-1}\xi(p)p^{k-1})R(k)c(dp^m)^2$$

Here $m = 0$ in case a, and $m = 1$ in case b.

Consider now the point $P = (2, id)$, and put

$$\int_{\mathbb{Z}_\Delta\times\Gamma}\xi_d(z)w^2d\mu(z,w) = \mathcal{L}(d).$$

The numbers $\mathcal{L}(d)$ belong to $\mathcal{O}_K$. Put $v = \inf_d v_p(\mathcal{L}(d)) \geq 0$. Notice that $\gamma = a_2(p)$ is 1 or -1 ([10], Theorem 4.6.17 (2)). If $\xi_d(p) = \gamma$, then the factor $1 - A_p(P)^{-1}\xi_d(p)^{-1}p^\nu$ vanishes in the right-hand side of [6]. In fact, $L(f_P,\xi_d,1)$ also vanishes at the same time because of the sign of the functional equation [9]. It follows that $\xi_d(p) = \gamma$ yields $\mathcal{L}(d) = 0$, and not all of these numbers are zero. In particular, $v < \infty$.

One has

$$v_p\left(\int_{\mathbb{Z}_\Delta\times\Gamma}\xi(z)z^{k-1}w^{2k-1}d\mu(z,w) - \int_{\mathbb{Z}_\Delta\times\Gamma}\xi_d(z)w^2d\mu(z,w)\right) \geq r$$

as soon as

$$v_p\left(\xi(z)z^{k-1}w^{2k-1} - \xi_d(z)w^2\right) \geq r \quad \text{for all} \quad z,w \in \mathbb{Z}_\Delta\times\Gamma. \tag{3}$$

This fact (abstract Kummer congruences) is discussed in [11], Chapter 1, in details.

Our setting yields (3). This is evident in case a. In case b, both $\xi(\cdot)$ and "$\xi(\cdot)\cdot^{k-1} \bmod p^r$" are nontrivial (since odd) quadratic characters of the cyclic group $\mathbb{Z}/\Delta p^r\mathbb{Z}$. It follows that they are equal. This yields (3) in case b.

Therefore we have

$$v_p\left(\alpha^{-1}(1-\xi(p)\beta p^{-k})(1-\alpha^{-1}\xi(p)p^{k-1})R(k)c(dp^m)^2 - \mathcal{L}(d)\right) \geq r.$$

This inequality yields the conclusion of the theorem as soon as $r \geq v$, because $R(k)$ does not depend on d.

Remarks.

1. One might also try to indicate the degree of congruences. For this purpose one introduces a number $t \geq 1$, and exchange the condition $v_p(c(d)) \geq 1$ in case a ($v_p(c(pd)) \geq 1$ in case b) in the definition by $v_p(c(d)) \geq t$ (resp. $v_p(c(pd)) \geq t$). These will be the quadratic congruences of degree t. In our notations, one can take $t = r$ in the case of Cohen-Eisenstein series, and $t = [(r-v)/2]$ in the case of cusp forms. The argument stays the same.

2. The absence of modulo p congruences, assumed in the theorem, is not the necessary condition. It is the existence of the two-variable p-adic L-function, which is crucial for our argument. More precisely, consider the universal ordinary Hecke algebra $\mathcal{R}$ [4]. This is a Λ-algebra, and, in general, $\mathcal{R}$ is larger then Λ. A newform corresponds to a $\bar{\mathbb{Q}}_p$-valued homomorphism of $\mathcal{R}$. The local consideration (see 2.7 of [4]) yields that the arithmetic points P in a neighborhood of $(2, id)$ parametrize an analytic family of the $\bar{\mathbb{Q}}_p$-valued homomorphisms of $\mathcal{R}$. Theorem 5.15 **a** of [4] provides us after that with a p-adic L-function in two variables. It seems plausible that the twisting with a quadratic Dirichlet character will not harm the construction. One can make use of this L-function instead of those of proposition 2. Starting with this point our argument can be continued without changes. However, it seems that some additional condition should be involved anyway. The current author can not reformulate this in the language which fits into our consideration. Another possibility is to make use of the construction of a Λ-adic Shintani lifting [12].

2. One can also impose an auxiliary conductor (i.e. consider the Λ-adic forms of level Np^∞ instead of p^∞ considered). This will cover the modulo 7 congruences found in [1]. The argument should be essentially the same.

3. This is highly probable that our $R(k)$ is almost always a p-adic unit. This number appears because there is no natural normalization for modular forms of half-integral weight. We consider $R(k)$ as a "random part" of the phenomena.

Note, that due to this part, we have not proven any singular congruence. We have proven the existence of infinite families of congruences instead. Our result predicts the couples (weight, prime number) when the quadratic congruences might appear. Acting, for instance, along the same lines as in [1], one can check the following examples with machine computation. In all these examples $\dim S_{2k} = 1$.

$k = 11$	$p = 11$	type a
$k = 9$	$p = 17$	type b
$k = 6$	$p = 11$	type b, presented in [1]

$$k = 10 \quad p = 19 \quad \text{type b}$$

One can easily check, however, that the congruences does not hold for $k = 13$. This is because the space $S_2(\Gamma_0(13))$ is empty. This, in its turn, coincides with the fact that the unique cusp form of weight 26 is not 13-ordinary.

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