

# QUADRATIC CONGRUENCES FOR COHEN - EISENSTEIN SERIES.

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The notion of quadratic congruences was introduced in the recently appeared paper [1]. In this note we present another, somewhat more conceptual proof of several results from loc. cit. Our method allows to refine the notion and to generalize the results quoted. Here we deal only with the quadratic congruences for Cohen - Eisenstein series. A similar phenomena exists for cusp forms of half-integral weight as well. However, as one can expect, in the case of Eisenstein series the argument is much simpler. In particular, we do not make use of other techniques than  $p$ -adic Mazur measure, whereas the consideration of cusp forms of half-integral weight involves much more sophisticated construction. Moreover, in the case of Cohen-Eisenstein series we are able to get the full and exhaustive result. For these reasons we dare to present the argument here.

Our result deals with modular forms, but the argument essentially does not. One can think just about the congruences for Cohen numbers  $H(r, N)$ . These are arithmetically interesting rational numbers defined below.

Our proof relies on the construction of  $p$ -adic Mazur measure. We formulate the precise statement as proposition 1 in the text. After that we present a corollary (proposition 2) which is sufficient for our purposes.

Let  $\chi$  be a Dirichlet character modulo  $M > 1$ , and denote by  $L(s, \chi)$  the associated  $L$ -series.

$$L(s, \chi) = \sum_{n \geq 1} \chi(n)n^{-s}.$$

The series converges for  $\Re(s) > 1$  and admits the analytic continuation over all  $s \in \mathbb{C}$ . Its values at negative integers essentially coincide with generalized Bernoulli numbers. More precisely, one has for a positive integer  $r$

$$L(1 - r, \chi) = -\frac{B_{r, \chi}}{r},$$

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where the numbers  $B_{r,\chi}$  are defined by

$$\sum_{a=1}^M \frac{\chi(a)te^{at}}{e^{Mt} - 1} = \sum_{r \geq 0} B_{r,\chi} \frac{t^r}{r!}.$$

Fix an integer  $r \geq 2$ . If  $N = 0$ , then let  $H(r, 0) = \zeta(1 - 2r)$ . Here  $\zeta$  denotes the Riemann  $\zeta$ -function. If  $N$  is a positive integer and  $Df^2 = (-1)^r N$ , where  $D$  is the discriminant of a quadratic field, then define  $H(r, N)$  by

$$(1) \quad H(r, N) = L(1 - r, \chi_D) \sum_{d|f} \mu(d) \chi_D(d) d^{r-1} \sigma_{2r-1}(f/d).$$

Here  $\chi_D$  denotes the quadratic character associated with  $\mathbb{Q}(\sqrt{D})$  [2]. The arithmetic function  $\sigma_{2r-1}$  is defined by  $\sigma_{2r-1}(l) = \sum_{d|l} d^{2r-1}$ . We denote by  $\mu$  the Möbius function. In particular, if  $D = (-1)^r N$  is the discriminant of a quadratic field, then

$$H(r, N) = L(1 - r, \chi_D) = -\frac{B_{r,\chi_D}}{r}.$$

In all other cases let  $H(r, N) = 0$ .

The rational numbers  $H(r, N)$  were introduced by H. Cohen [3]. Their significance is connected with the following fact. The series

$$\mathcal{H}_r = \sum_{n \geq 0} H(r, N) \exp(2\pi i N z)$$

is the Fourier expansion of a modular form of half-integral weight  $r + 1/2$ . This is the Cohen - Eisenstein series.

Since we are going to deal with congruences, it will be convenient to introduce some basic notations from  $p$ -adic analysis. Let  $p$  be a prime. The restriction  $p \neq 2$  will somewhat simplify our argument, and we always assume it. Denote by  $\mathbb{Q}_p$  the field of  $p$ -adic numbers, i.e. the completion of the field of rational numbers  $\mathbb{Q}$  with respect to the  $p$ -adic metric given by the  $p$ -adic valuation

$$\begin{aligned} |\cdot|_p : \mathbb{Q} &\rightarrow \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\} \\ |a/b|_p &= p^{\text{ord}_p b - \text{ord}_p a}, \quad |0|_p = 0, \end{aligned}$$

where  $v_p(a) = \text{ord}_p(a)$  is the highest power of  $p$  dividing the integer  $a$ . We fix an embedding  $\iota : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ , and we will not make difference between rational numbers and their images under  $\iota$ . This allows us to consider rational numbers as  $p$ -adic as well.

Let us now define the quadratic congruences.

**Definition.** Let  $p$  be an odd prime. Let  $\varphi = \sum_{n \geq 0} a(n) \exp(2\pi i n z)$  be a modular form of half-integral weight  $r + 1/2$ . Assume that the Fourier

coefficients  $a(n)$  are rational numbers. For a positive integer  $d$ , denote by  $\xi_d$  the quadratic Dirichlet character associated with  $\mathbb{Q}(\sqrt{-d})$ .

**a** We say that  $\varphi$  satisfies type a quadratic congruences modulo  $p$  if  $r - 1 \equiv 0 \pmod{(p - 1)p^{\alpha - 1}}$  for an integer  $\alpha \geq 1$ , and

$$\xi_d(p) = 1 \quad \text{yields} \quad v_p(a(d)) \geq \alpha.$$

**b** We say that  $\varphi$  satisfies type b quadratic congruences modulo  $p$  if  $k - 1 \equiv 0 \pmod{(p - 1)p^{\alpha - 1}/2}$  for an integer  $\alpha \geq 1$ ,  $k - 1 \not\equiv 0 \pmod{p - 1}$ , and

$$\xi_d(p) = 1 \quad \text{yields} \quad v_p(a(pd)) \geq \alpha.$$

### Remarks.

1. Our definition is sufficient since we are going to consider only Cohen - Eisenstein series. In general, one should consider cusp forms of half-integral weight as well. Let us indicate, without explaining the details, the necessary changes in the definition above. One should assume  $\varphi$  to be a Hecke eigenform [7]. Then  $a(n)$  are no more rational but algebraic numbers. We extend our fixed embedding  $\iota$  to an embedding  $\iota : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ . Here  $\bar{\mathbb{Q}}$  is the algebraic closure of the field of rational numbers, and  $\mathbb{C}_p$  is the Tate field (i.e. the completion of the algebraic closure of  $\mathbb{Q}_p$  [6]). One also should sometimes change the condition  $\xi_d(p) = 1$  for  $\xi_d(p) = -1$ .

2. Our definition is slightly different from the original one given in [1]. After the changes described in the remark above, all the quadratic congruences for the Fourier coefficients of modular forms of half-integral weight presented in loc.cit. will fit into our definition.

The main result of the present note is the following assertion.

**Theorem.** *The Cohen-Eisenstein series  $\mathcal{H}_r$  satisfy both a and b types modulo  $p$  quadratic congruences as soon as  $r - 1 \equiv 0 \pmod{(p - 1)/2}$ .*

Several examples of this phenomena are described in Section 2 of [1]. To be more specific, put  $r = 5$  and  $p = 5$ . Theorem 6 of [1] asserts that  $N \equiv 1 \pmod{5}$  yields  $H(5, N) \equiv 0 \pmod{5}$ . This is (half of) our type a quadratic congruences. The original proof of the theorem quoted is based on the fact that  $\mathcal{H}_r$  is a modular form. The argument uses the knowledge of specific structure of the space of modular forms of weight 6 on  $\Gamma_1(500)$  and involves a machine computation.

**Proof.** Recall the construction of the  $p$ -adic Mazur measure. We address the reader to [10], Chapter 1; [11], Chapters 5,7,12; [4], Chapter 3, where detailed discussions and various interpretations are given. The first original approach is contained in [8], [9].

**Proposition 1.** *Let  $\Delta$  be a positive integer,  $(\Delta, p) = 1$ . Put*

$$\mathbb{Z}_\Delta = \varprojlim_m (\mathbb{Z}/\Delta p^m \mathbb{Z})^*.$$

*Let  $c > 1$  be an integer coprime to  $\Delta$  and  $p$ .*

*There exists a (integer-valued) measure  $\mu_{c,\Delta}$  on  $\mathbb{Z}_\Delta$  such that*

$$(2) \quad \int_{\mathbb{Z}_\Delta} \xi(a) a^{r-1} d\mu_{c,\Delta} = -(1 - \xi(c)c^r)(1 - \xi(p)p^{r-1})L(1 - r, \xi).$$

The following corollary is a special case of *abstract Kummer congruences* [10], Chapter 3; [5], p.258. In order to get it, one notices that the integrals of ( $p$ -adically) close functions against a compact set ( $\mathbb{Z}_\Delta$  is compact) are ( $p$ -adically) close.

**Proposition 2.** *Let  $\Delta$  be a positive integer,  $(\Delta, p) = 1$ . Let  $c > 1$  be an integer coprime to  $\Delta$  and  $p$ . Let  $\xi_i$  for  $i = 1, 2$  be two Dirichlet characters modulo  $\Delta p^{m_i}$  for integers  $m_i \geq 0$ , and  $r_i$  two positive integers. Suppose that for every  $a$  coprime to  $\Delta$  and  $p$  one has*

$$(3) \quad \xi_1(a) a^{r_1-1} \equiv \xi_2(a) a^{r_2-1} \pmod{p^\alpha}, \quad \alpha \geq 0.$$

*Then the right hand sides of (2) for  $i = 1, 2$  are also congruent modulo  $p^\alpha$ :*

$$(4) \quad v_p((1 - \xi_1(c)c^{r_1})(1 - \xi_1(p)p^{r_1-1})L(1 - r_1, \xi_1) - (1 - \xi_2(c)c^{r_2})(1 - \xi_2(p)p^{r_2-1})L(1 - r_2, \xi_2)) \geq \alpha.$$

We apply proposition 2 to the case when both  $\xi_i$  are quadratic Dirichlet characters. In particular, we take the absolute value of the discriminant of  $\mathbb{Q}(\sqrt{-d})$  for  $\Delta$ , the quadratic Dirichlet character  $\xi_d$  associated with  $\mathbb{Q}(\sqrt{-d})$  for  $\xi_1$ , and put  $r_1 = 1$ . Observe that if  $\xi_d(p) = 1$ , the right-hand side of 2 vanishes because of the factor  $(1 - \xi_1(p)p^{r_1-1})$ . Put  $r_2 = r$ .

Let us now pick appropriate  $\xi_2$ .

**a.** Assume  $r - 1 \equiv 0 \pmod{(p-1)p^\alpha}$ . Put  $\xi_2 = \xi_d$ . The condition (3) is fulfilled. Pick the number  $c$  such that  $1 - \xi_d(c)c^r \equiv 1 - \xi_d(c)c \not\equiv 0 \pmod{p}$ . Therefore by (4)  $v_p(L(1 - r, \xi)) \geq \alpha$ . It now follows from (1) that  $\mathcal{H}_r$  satisfies the type a quadratic congruences.

**b.** Assume  $r - 1 \equiv 0 \pmod{(p-1)p^\alpha/2}$  and  $r - 1 \not\equiv 0 \pmod{(p-1)}$ . Take the quadratic Dirichlet character associated with  $\mathbb{Q}(\sqrt{(-1)^r dp})$  for  $\xi_2$ . Notice that since  $-d \equiv (-1)^r dp \pmod{4}$ , the character  $\xi_2$  is defined modulo  $\Delta p$  correctly. We claim that for every  $a$  coprime to  $\Delta$  and  $p$

$$\xi_d(a) \equiv \xi_2(a) a^{r-1} \pmod{p^\alpha}.$$

Indeed, both  $\xi_d(\cdot)$  and " $\xi_2(\cdot) \cdot r^{-1} \bmod p^\alpha$ " are non-trivial (since both characters are odd:  $\xi_d(-1) \equiv \xi_2(-1)(-1)^{r-1} = -1$ ) quadratic characters of the multiplicative group  $(\mathbb{Z}/\Delta p^\alpha \mathbb{Z})^*$ . Since this group is cyclic, they are equal. It follows now from (1) that  $\mathcal{H}_r$  satisfies the type b quadratic congruences.

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