RATIONALITY OF THE FOLSOM - ONO GRID

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ABSTRACT. In a recent paper Folsom and Ono constructed a grid of Poincaré series of weights $3/2$ and $1/2$. They conjectured that the coefficients of the holomorphic parts of these series are rational integers. We prove that these coefficients are indeed rational numbers with bounded denominators.

1. INTRODUCTION

In a recent paper [9] Folsom and Ono constructed two collections of Poincaré series of weight $1/2$ (resp. $3/2$) numerated by non-positive integers m (resp. negative integers.) These series are weak harmonic Maass forms (resp. weakly holomorphic modular forms), and the q-expansion coefficients of their holomorphic parts satisfy a surprising duality identity. This fact constitutes the principal result of [9], and confirms that the families of Poincaré series, constructed by Folsom and Ono, are very natural and interesting. This duality result is not isolated: similar kind of grids were recently constructed in various setting (e.g. [15, 3, 8, 11]). The setting considered by Folsom and Ono in [9] is of special interest, because the holomorphic part of their weight $1/2$ Poincaré series which corresponds to $m = 0$ coincides essentially with the Ramanujan mock theta function

$$f(q) := 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}. \tag{1}$$

This function obviously has integer q-expansion coefficients, moreover, computer calculations suggest that this phenomenon holds for other m. Based on this observation, Folsom and Ono conjectured in [9] that all the q-expansion coefficients involved are rational integers. As a clue, how a somehow weaker statement may be proved, they suggested (see Remark 3 to Theorem 1.1 in loc. cit.) to employ the Galois action on $M^!_{3/2}\left(144, \left(\frac{12}{\cdot}\right)\right)$, and the fact that $S_{3/2}\left(144, \left(\frac{12}{\cdot}\right)\right) = 0$. Unfortunately this hint contains a typo (in fact, $\dim S_{3/2}\left(144, \left(\frac{12}{\cdot}\right)\right) = 2$, see [7]), which seemingly completely compromises it. However, their idea is correct and powerful enough. In this paper we develop this idea, and make use of the emptiness of the space $S_{1/2}\left((144, \left(\frac{12}{\cdot}\right)\right)$ along with the one-dimensionality of the space $S_{3/2}\left(36, \left(\frac{12}{\cdot}\right)\right)$ in order to prove the following result.

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Theorem 1. *Let for an integer* $m \leq 0$

$$H^+_{0,m} = q^{-1/24}\left(q^m + \sum_{n\geq 0} a_m(n)q^n\right)$$

be the q-expansion of the holomorphic part of the weight $1/2$ *Poincaré series constructed by Folsom and Ono. Then all numbers* $a_m(n)$ *are rational.*

Remark 1. The question on rationality (algebraicity) of the Fourier coefficients of holomorphic parts of weak harmonic Maass forms may be deep and subtle. In particular, the proof of the rationality in a certain integral weight situation [6] is quite non-trivial. On the opposite side, conjectural irrationality in another integral weight setting would imply famous Lehmer's conjecture [13]. Most interestingly, the rationality of particular coefficients in a half-integral weight setting in [5] allows Bruinier and Ono to conclude that a certain elliptic curve has an infinite rang, and to interpret the rational coefficients in terms of the central derivatives of its L-function.

Recall that Folsom and Ono conjectured in [9] that the numbers $a_m(n)$ are in fact rational *integers*. In this direction, we prove the following statement.

Corollary 1. *For every* $m \leq 0$ *the denominators of* $a_m(n)$ *are bounded independently on* n.

Remark 2. As it was recently communicated to the author, Sander Zwegers has succeeded to use the results of the present paper in order to prove that all coefficients $a_m(n)$ are indeed rational integers. He does that by means of an explicit construction, which, in particular, provides an alternative independent way of calculation these coefficients. The coefficients calculated by Zwegers agree totally with those calculated by Folsom and Ono. The paper [9] contains a typo. The author thanks Ken Ono for communicating a correct version of the numerics. We display it below (in our notations).

$$\begin{aligned} H^+_{0,0} &= q^{-1/24}\left(1 + q - 2q^2 + 3q^3 - 3q^4 + \ldots + 486q^{47} + \ldots\right) \\ H^+_{0,-1} &= q^{-1/24}\left(q^{-1} - 263q + 2781q^2 - 17960q^3 + \ldots\right) \\ H^+_{0,-2} &= q^{-1/24}\left(q^{-2} + 3400q - 102060q^2 + \ldots\right) \\ H^+_{0,-3} &= q^{-1/24}\left(q^{-3} - 23374q + \ldots\right) \end{aligned}$$

We review some facts on vector-valued weak harmonic Maass forms in Section 2, and the construction of vector-valued Maass - Poincaré series in Section 3 of the paper. Section 4 is devoted to the proofs of Theorem 1 and Corollary 1.

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2. Preliminaries on vector-valued weak harmonic Maass forms

In this section we introduce notations and recall some basic facts on vector-valued weak harmonic Maass forms. We closely follow the exposition in [4], and refer to this paper for a detailed discussion. The only discrepancy between our notations and that of [4] is that we allow an arbitrary unitary representation ρ, not necessarily the Weil representation as the Nebentypus.

Let $Mp_2(\mathbb{R})$ be the two-fold cover of $SL_2(\mathbb{R})$, realized by the two choices of the holomorphic square roots of $j(\gamma,\tau)=c\tau+d$, where $\Im\tau>0$, and $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{R})$. Recall that the elements of $Mp_2(\mathbb{R})$ are of the form $(\gamma,\phi(\tau))$ with $\gamma\in SL_2(\mathbb{R})$ and a holomorphic function $\phi(\tau)$ such that $\phi(\tau)^2=j(\gamma,\tau)$. For a complex number $z=|z|\exp(i\theta)$ with $\theta\in(-\pi,\pi]$ we assume $\sqrt{z}=z^{1/2}=|z|^{1/2}\exp(i\theta/2)$.

We denote by Γ the inverse image of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{R})$, and denote by

$$\tilde{S}=\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\sqrt{\tau}\right) \quad\text{and}\quad \tilde{T}=\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},1\right)$$

the standard generators of Γ. Let ρ be an r-dimensional unitary representation of Γ. Let $k\in\frac{1}{2}\mathbb{Z}$. We define a weak harmonic Maass form of weight k and Nebentypus ρ as a real-analytic function from the complex upper half-plane to $\mathbb{C}^r$ which satisfies the following conditions:

i) $f(\gamma\tau)=\phi(\tau)^{2k}\rho(\gamma,\phi)f(\tau)$ for all $(\gamma,\phi)\in\Gamma$.

ii) there is a $C>0$ such that for any cusp $s\in\mathbb{Q}\cup\{\infty\}$ of $SL_2(\mathbb{Z})$ and $(\gamma,\phi)\in Mp_2(\mathbb{R})$ with $\gamma\in SL_2(\mathbb{Z})$ such that $\gamma\infty=s$, the function $f_s(\tau)=\phi^{-2k}\rho^{-1}(\gamma,\phi)f(\gamma,\tau)$ satisfies $f_s(\tau)=\mathcal{O}(\exp(Cv)$ as $v\to\infty$ (uniformly in u, where $\tau=u+iv$).

iii) $\Delta_k f(\tau)=0$, where

$$\Delta=-v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+ikv\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right).$$

We denote the space of harmonic weak Maass forms of weight k and Nebentypus ρ by $H_{k,\rho}$. Many properties of our weak harmonic Maass form coincide with those described in [4, Section 3] with literally the same proofs since these proofs do not use any specifics about the Weil representation. In particular, any weak harmonic Maass form f of weight k (we think of f as a column vector of r functions) has a unique decomposition $f=f^++f^-$ into its holomorphic and non-holomorphic parts with Fourier expansions

$$f^+(\tau)=\sum_{n\in\mathbb{Q}}a^+(n)\exp(2\pi inu), \tag{2}$$

and

$$f^-(\tau)=a^-(0)v^{1-k}+\sum_{n\in\mathbb{Q},\ n\neq 0}a^-(n)H(2\pi nv)\exp(2\pi inu), \tag{3}$$

where

$$H(w)=e^{-w}\int_{-2w}^{\infty}e^{-t}t^{-k}dt.$$

All but finitely many coefficients $a^+(n)$ with $n<0$ (resp. $a^-(n)$ for $n>0$) in (2) (resp. (3) vanish. If $f^-=0$, then f is a vector valued weakly holomorphic modular form of weight k and Nebentypus ρ. (Note that f may have fractional powers of q in its q-expansion.) We denote the space these forms by $M^!_{k,\rho}$. We denote by $M_{k,\rho}$ and $S_{k,\rho}$ the subspaces of modular (bounded at cusps) and cups (vanishing at cusps) forms.

The operator $\xi : H_{k,\rho} \to M^!_{2-k,\bar\rho}$ is defined by

$$\xi := -2iv^k \overline{\frac{\partial}{\partial\bar\tau}}.$$

Its action on $f \in H_{k,\rho}$ is given by

$$\xi(f) = \xi(f^-) = -2v^k \overline{(k-1)a^-(0) + \sum_{n\in\mathbb{Q},\ n\neq 0} a^-(n)(-4\pi n)^{1-k}\exp(2\pi n\bar\tau)}. \tag{4}$$

The inverse image of $S_{2-k,\rho}$ under ξ is denoted by $H^+_{k,\rho}$.

The bilinear pairing between the spaces $S_{2-k,\bar\rho}$ and $H^+_{k,\rho}$ is defined by

$$\{g,f\} = \text{constant term of the } q\text{-series} \quad < g, f^+ >$$

The argument in the proof of [4, Proposition 3.5] remains unaltered, and proves

$$\{g,f\} = (g, \xi(f)) \tag{5}$$

with the Petersson scalar product in the right-hand side.

3. Vector valued Maass - Poincaré series

In this section we provide a construction of a collection of vector valued weak harmonic Maass forms making use of Poincaré series. For an integer l we use the notation $\zeta_l := \exp(2\pi i/l)$. In order to introduce the Poincaré series constructed by Folsom and Ono [9] we need additional notations.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ with $c > 0$ put

$$\chi(\gamma) := \begin{cases} \zeta_{24}^{-b} & c = 0 \\ i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)}\exp(2\pi i(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}))\omega^{-1}_{-d,c} & c > 0, \end{cases}$$

where

$$\omega_{d,k} := \exp(\pi i s(d,k)),$$

and the Dedekind sum $s(d,k)$ is defined by

$$s(d,k) := \sum_{j \mod k} \left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{dj}{k}\right)\right),$$

with

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & x \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

We now define the weight 1/2 Poincaré series following [9] for $m \le 0$

$$H_{0,m}(\tau) := \frac{2}{\sqrt{\pi}} \sum_{\gamma\in\Gamma_0(2)/\pm\Gamma_\infty} \chi(\gamma)^{-1}(c\tau+d)^{-1/2}\phi_{1/2,3/4}(m - 1/24, \gamma\tau),$$

where γ is represented by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here the function $\phi_{\kappa,s}(r,\tau)$ is defined by the M-Whittaker function $M_{\nu,\mu}$ for $v \geq 0$:

$$\phi_{\kappa,s}(r,z) := (4\pi|r|v)^{-\kappa/2}\exp(2\pi iru)M_{-\kappa/2,s-1/2}(4\pi|r|v).$$

Along with $H_{0,m}$ we define the following two companion series (cf. [10]):

$$H_{1,m}(\tau) := \frac{2\sqrt{-i}}{\sqrt{\pi}} \sum_{\gamma\in\Gamma_0(2)/\pm\Gamma_\infty} \chi(\gamma)^{-1}(d\tau - c)^{-1/2}\phi_{1/2,3/4}(m-1/24, \gamma S\tau),$$

and

$$H_{2,m}(\tau) := \frac{2\sqrt{-i}\zeta_3}{\sqrt{\pi}} \sum_{\gamma\in\Gamma_0(2)/\pm\Gamma_\infty} \chi(\gamma)^{-1}(d\tau - (d-c))^{-1/2}\phi_{1/2,3/4}(m-1/24, \gamma ST\tau),$$

where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Let ρ be the three-dimensional representation of Γ defined on the generators by

$$\rho(\tilde{S}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \rho(\tilde{T}) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix}$$

We collect all we need for further applications about these Poincaré series in the following proposition.

Proposition 1. (i) *For every* $m \leq 0$

$$H_m := (H_{0,m}, H_{1,m}, H_{2,m})^T \in H_{1/2,\rho}$$

(ii) *We have the Fourier expansions*

$$\begin{aligned} H_{0,m}^+ &= q^{m-1/24} + \mathcal{O}(q^{23/24}) \\ H_{1,m}^+ &= \mathcal{O}(q^{1/3}) \\ H_{2,m}^+ &= \mathcal{O}(q^{1/3}) \end{aligned}$$

Sketch of the proof. The series $H_{1,m}$ and $H_{2,m}$ are constructed in a way such that condition i) of the definition of weak harmonic Maass forms of weight 1/2 with representation ρ is satisfied automatically (see [10] for a similar construction). After that one computes the Fourier coefficients of all the three series using Poisson summation (cf. [9, Lemma 3.3]), and then justifies the convergence of the coefficients of the expressions obtained for Fourier coefficients. This argument is provided in full details in [9] (Theorems 3.1 and 3.3) for the series $H_{0,m}$; the arguments for the series $H_{1,m}$ and $H_{2,m}$ are similar. As soon as the Fourier expansions are calculated, (ii) becomes clear, and it is easy to check that $H_m \in H_{1/2,\rho}$. We further have to show that the absolute values of the Fourier coefficients of $\xi(H_m)$ grow polynomially with n. In view of (4), this follows from a similar estimate for the Fourier coefficients of the non-holomorhic part of H_m. In the special case of $H_{0,0}$, this is done by Bringmann and Ono in [2, Corollary 4.2]. The general case is simlar to that. The polynomial growth of these Fourier coefficients implies $\xi(H_m) \in M_{3/2,\bar{\rho}}$. However, the transformation law with respect to $\tilde{T}$ shows that neither one of the three series may have a non-zero constant term. This observation implies $\xi(H_m) \in S_{3/2,\bar{\rho}}$ completing the proof of the proposition.

$\square$

4. Proofs of Theorem 1 and Corolary 1

Proposition 2. *The space $S_{3/2,\bar{\rho}}$ is one-dimensional.*

Proof. Following Zwegers [16] we consider the theta functions

$$\begin{aligned} g_0(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3) \exp(3\pi i (n+1/3)^2 \tau) \\ g_1(\tau) &= -\sum_{n\in\mathbb{Z}} (n+1/6) \exp(3\pi i (n+1/6)^2 \tau) \\ g_2(\tau) &= \sum_{n\in\mathbb{Z}} (n+1/3) \exp(3\pi i (n+1/3)^2 \tau). \end{aligned}$$

The transformation law for these theta-functions, proved by Zwegers in [16], tells us that $\Phi := (g_1, g_0, -g_2)^T \in S_{3/2,\bar{\rho}}$. Note that

$$3g_2(6\tau) = \frac{1}{2} \sum_{n\in\mathbb{Z}} \left(\frac{-3}{n}\right) n q^{n^2} \in S_{3/2}\left(\Gamma_0(36), \left(\frac{12}{\cdot}\right)\right)$$

It is easy to check (see e.g. [7]) that $\dim S_{3/2}\left(\Gamma_0(36), \left(\frac{12}{\cdot}\right)\right) = 1$. At the same time if $(h_1, h_0, -h_2) \in S_{3/2,\bar{\rho}}$, then, since the transformation law of all three elements is determined completely, we must have $h_2 = cg_2$ with a constant c. This immediately implies $h_0 = cg_0$ and $h_1 = cg_1$, which proves the proposition.

$\square$

Proposition 3. *For an integer $m \le 0$, let*

$$C_m = \begin{cases} \left(\frac{-6}{n}\right) n & \text{if } 1-24m = n^2 \text{ for an integer } n \ge 1 \\ \\ 0 & \text{otherwise.} \end{cases}$$

The function $H_m - C_m H_0$ is a vector valued weakly holomorphic modular form of weight $1/2$:

$$F_m := H_m - C_m H_0 \in M^!_{1/2,\rho}.$$

Proof. By construction (see Proposition 1) $F_m \in H_{1/2,\rho}$. Thus all we have to check is $F_m^- = 0$, that is $\xi(F_m) = 0$. We employ (5) in order to calculate $\xi(H_m)$. By Proposition 2 there is a $c_m \in \mathbb{C}$ such that $\xi(H_m) = c_m \Phi$. Note that

$$g_1(\tau) = -\frac{1}{6} \sum_{n\ge 1} \left(\frac{-6}{n}\right) n q^{n^2/24},$$

and the constant term of the q-expansion of $g_0 H^+_{1,m} - g_2 H^+_{2,m}$ vanishes for every $m \le 0$. It follows that for all $m \le 0$

$$\begin{aligned} c_m(\Phi,\Phi) &= \{\Phi, H_m\} = \text{constant term of } g_1 H^+_{0,m} \\ &= \begin{cases} -\dfrac{1}{6}\left(\dfrac{-6}{n}\right) n & \text{if} \qquad m = \dfrac{1-n^2}{24} \\ \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus $c_m = C_m c_0$, and

$$\xi(F_m) = \xi(H_m - C_m H_0) = c_m \Phi - C_m c_0 \Phi = 0$$

as required. □

Since, by Proposition 3, $F_m(\tau) \in M^!_{1/2,\rho}$, we derive with an argument identical to the proof of [2, Corollary 2.3] that

$$F_{0,m}(24\tau) = H_{0,m}(24\tau) - C_m H_{0,0}(24\tau) \in M^!_{1/2}\left(\Gamma_0(144), \left(\frac{12}{\cdot}\right)\right).$$

Note also that $C_m H^+_{0,0}(24\tau)$ has rational integral Fourier coefficients since C_m is an integer by definition, and by the prominent result of Bringmann and Ono [2] $H^+_{0,0} = q^{-1/24} f(q)$, where $f(q)$ is the Ramanujan mock theta-function (1), which manifestly has rational integral Fourier coefficients. The rationality of Fourier coefficients of $H^+_{0,m}$, claimed in Theorem 1 and the boundedness of their denominators, is thus equivalent to the same properties of Fourier coefficients of $F_{0,m}$, which we are going to prove now.

Proposition 1 implies that function $F_{0,m}(\tau)$, which has a pole at infinity, has a pole at every cusp c/d with $g.c.d.(c,d) = 1$ and $2|d$, and has a zero at all other cusps. It follows that the weakly holomorphic modular form $F_{0,m}(24\tau)$ has a pole at infinity and every cusp c/d with $g.c.d.(c,d) = 1$ and $48|d$, and has a zero at all other cusps. Note that the principal part of the Fourier expansion of $F_{0,m}(24\tau)$ at infinity is a polynomial in q^{-1} with rational coefficients. Let

$$\eta(\tau) = q^{1/24} \prod_{n\geq 1}(1-q^n)$$

be Dedekind's eta-function. We now introduce two auxiliary eta-quotients (cf. [12, Theorem 1.64]):

$$\Phi(\tau) := \frac{\eta(8\tau)\eta(48\tau)^8\eta(72\tau)^3}{\eta(16\tau)^2\eta(24\tau)^4\eta(144\tau)^6} \in M_0\left(\Gamma_0(144), \xi_0\right),$$

where ξ_0 denotes the trivial character modulo 144, and

$$\Psi(\tau) := \frac{\eta(24\tau)\eta(144\tau)^6}{\eta(48\tau)^2\eta(72\tau)^3} \in M_1\left(\Gamma_0(144), \left(\frac{-3}{\cdot}\right)\right).$$

It follows from [12, Theorem 1.65] that eta-quotient $\Phi(\tau)$ has a pole at infinity, a positive order of vanishing at the cusp 1/48 (i.e. c/d with $g.c.d.(c,d) = 1$ and $g.c.d.(144,d) = 48$), and zero order of vanishing at all the other cusps of $\Gamma_0(144)$. Thus for a big enough positive integer A

$$F_{0,m}(24\tau)\Phi(\tau)^A \in M^!_{1/2}\left(\Gamma_0(144), \left(\frac{12}{\cdot}\right)\right).$$

has no other poles than at infinity and a positive order of vanishing at all the other cusps of $\Gamma_0(144)$. It follows from [12, Theorem 1.65] that eta-quotient $\Psi(\tau)$ has a zero at infinity, and zero order of vanishing at all the other cusps of $\Gamma_0(144)$. Thus for a big enough positive integer B

$$f := F_{0,m}(24\tau)\Phi(\tau)^A\Psi(\tau)^{2B-1} \in S_{2B-1/2}\left(\Gamma_0(144), \left(\frac{-12}{\cdot}\right)\right).$$

We now need the following proposition, which is a consequence and a half-integral weight analog of a result of Shimura [14, Theorem 3.52].

Proposition 4. *For positive integers B and N with $4|N$ and a quadratic character ξ the space $S_{2B-1/2}(\Gamma_0(4N), \xi)$ has a basis consisting of cusp forms of which the Fourier coefficients at infinity are rational integers.*

Remark 3. The author is grateful to Prof. Skoruppa for a hint on the proof of this proposition. As it was communicated to the author, the proposition stays in fact true in a more general setting. In particular, the condition $4|N$ is redundant, and the statement holds for any positive half-integral weight. In this general setting, however, the proof requires a way of construction the half-integral weight modular forms spaces recently invented by Skoruppa.

Proof. Using the classical properties of the theta functions

$$\theta_2(\tau) = \sum_{m\geq 0} q^{(2m+1)^2} \quad \text{and} \quad \theta_3(\tau) = 1 + 2\sum_{m>0} q^{m^2},$$

it is easy to show (see [1, Chapter 4.2] for details) that under the conditions of Proposition 4 the map

$$S_{2B-1/2}(\Gamma_0(4N), \xi) \to S_{2B}(\Gamma_0(4N), \xi) \times S_{2B}(\Gamma_0(4N), \xi)$$

which takes f to the pair $(f\theta_2, f\theta_3)$ is an isomorphism of the space $S_{2B-1/2}(\Gamma_0(4N), \xi)$ to the subspace $U \subset S_{2B}(\Gamma_0(4N), \xi) \times S_{2B}(\Gamma_0(4N), \xi)$ which consists of the pairs (f_2, f_3) satisfying the condition $f_2\theta_3 = f_3\theta_2$. It follows from [14, Theorem 3.52] that the space $S_{2B}(\Gamma_0(4N), \xi)$ admits a basis with required properties. This fact implies the claim of Proposition 4. ☐

We are now in a position to finish the proof of Theorem 1. Let $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$, and denote by f^σ the series obtained by applying σ to the Fourier coefficients of f at infinity. It follows from Proposition 4 that $f^\sigma \in S_{2B-1/2}\left(\Gamma_0(144), \left(\frac{-12}{\cdot}\right)\right)$. Since the principal part of the Fourier expansion of $F_{0,m}(24\tau)$ (and, therefore, by construction, of f) has rational coefficients, and $\Psi(\tau)$ does not vanish at cusps other than infinity,

$$\frac{f - f^\sigma}{\Psi^{2B-1}}(\tau) = (F_{0,m}(24\tau) - F^\sigma_{0,m}(24\tau))\Phi(\tau)^A \in S_{1/2}\left(\Gamma_0(144), \left(\frac{12}{\cdot}\right)\right).$$

However, this space is empty: $\dim S_{1/2}\left(\Gamma_0(144), \left(\frac{12}{\cdot}\right)\right) = 0$, and it follows that $F_{0,m}(24\tau) = F^\sigma_{0,m}(24\tau)$, for every $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$, that is $F_{0,m}(24\tau) \in \mathbb{Q}[[q^{-1}, q]]$, and Theorem 1 is proved. ☐

Proof of Corollary 1. Since $F_{0,m}(24\tau)$ has rational Fourier coefficients, the same is true about f, and Proposition 4 implies that the denominators of the Fourier coefficients of f are bounded. Since $F_{0,m}(24\tau) = f(\tau)\Phi(\tau)^{-A}\Psi(\tau)^{1-2B}$, and the q-expansion

$$\Phi(\tau)^{-A}\Psi(\tau)^{1-2B} \in q^{16A+24(1-2B)}(1 + \mathbb{Z}[[q]]),$$

the denominators of the Fourier coefficients of $F_{0,m}$ are bounded for every $m \leq 0$. $\square$

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