p-ADIC INTERPOLATION OF TAYLOR COEFFICIENTS OF MODULAR FORMS

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ABSTRACT. In this paper we show that the Taylor coefficients of a Hecke eigenform at a CM-point, suitably modified, form a sequence of algebraic numbers that satisfy the Kubota - Leopoldt generalization of the Kummer congruences for primes that split in the imaginary quadratic field associated with a CM-point. More generally, we show that these numbers are moments of a certain p-adic measure. In addition, we write down explicitly the "Euler factor" at p in terms of the p-th Hecke eigenvalue of the modular form in question and certain data of the CM-point.

0. Introduction

For integers $n > 1$ define the Bernoulli numbers B_n via the Laurent expansion

$$\frac{1}{1-e^t} = \frac{1}{t} - \frac{1}{2} + \sum_{n>0} \frac{B_{n+1}}{n+1} \frac{t^n}{n!}.$$

The Kubota - Leopoldt generalization of the classical Kummer congruences for Bernoulli numbers can be stated as follows.

Theorem 0 *Let p be a prime and A a non-negative integer, and let n_1 be a positive integer such that $n_1 \not\equiv 1 \bmod p-1$. Then for any positive integer n_2 such that $n_1 \equiv n_2 \bmod (p-1)p^A$,*

$$E(n_1)\frac{B_{n_1}}{n_1-1} \equiv E(n_2)\frac{B_{n_2}}{n_2-1} \bmod p^{A+1}.$$

Here $E(n) = 1-p^n$ is the Euler factor at p for $\zeta(-n)$ where $\zeta(s)$ is the Riemann ζ-function.

In this paper we present another sequence of algebraic numbers $\{b_n\}$ that satisfy Kummer-type congruences. This sequence is closely related to the sequence of Taylor coefficients of a Hecke eigenform at a CM-point in the upper half-plane.

Today one views sequences satisfying Kummer-type congruences in terms of a p-adic measure on $\mathbb{Z}_p^*$ with prescribed moments (Mazur) or, equivalently, in terms of interpolation of a p-adic analytic function (Iwasawa). We adopt the former point of view in this paper.

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Our method depends on the ideas and techniques developed by Katz in [5]. In particular, Lemma 1 below can be viewed as a version of the q-expansion principle, and its proof is based on the constructions introduced in [5].

This paper is organized as follows. We formulate our result in Section 1. Section 2 is devoted to the connection between a certain differential operator ∂ and our modified Taylor coefficients. Section 3 contains a brief description of Katz's work [5]. In order not to reproduce a significant part of Katz's lengthy paper, we adopt some of his notations and make precise references to [5] for constructions and definitions. Our proof requires that the CM-point at which the Taylor expansion takes place is "suitable." We establish existence and give a complete description of suitable points in Section 4. The action of Hecke operators that allows us to extract the "Euler factor" is the subject of Section 5. Finally, by putting the results of Sections 2-5 together, we give a proof Theorems 1 in Section 6.

It has been pointed out to the authors that our result is closely connected with p-adic intepolation of square roots of central special values of a certain L-function. Harris and Tilouine write in the introduction of [3] that it should be possible to use Waldspurger's results in [11] (see also [2] in this connection) to interpolate these square roots p-adically. However, the authors were unable to establish an explicit link between the quantities b_n^2 and central special values of an L-function.

p-adic interpolation properties of the quantities b_n that we study were also studied by Mori [8]. In a recent preprint Mori partially carries out the suggestion of Harris, reports an error in [8] and presumably corrects it. Our setting is less general than that of A. Mori. This allows us to construct a $\mathbb{Z}_p^*$ measure (rather than only a $\mathbb{Z}_p$-measure that Mori obtains) and to write down the factors $\mathcal{E}(n)$ in Theorem 1 explicitly.

The second author is grateful to B. Mazur for a long, encouraging and enlightening discussion.

1. Statement and discussion of results. Let $f = \sum_n a(n)\exp(2\pi i n\tau)$ be a modular form on the full modular group $SL(2,\mathbb{Z})$ of even weight k. Assume f to be a Hecke eigenform, and let λ be the eigenvalue of the p-th Hecke operator T_p. Assume also that f is normalized so that all its Fourier coefficient $a(n)$ are algebraic integers.

Let ζ be a point in the complex upper-half plane which belongs to an imaginary quadratic field. Transform the upper-half plane onto a unit circle with the Möbius transform

$$z = \frac{\tau - \zeta}{\tau - \bar{\zeta}} \quad \tau \in \mathfrak{H}, \quad \text{the complex upper half-plane,}$$

so that the point ζ falls into the center of the circle. Here and everywhere in the paper bar denotes the complex conjugation. The modular form f becomes a complex-analytic function of z in the the unit circle. Write the Taylor expansion

$$\left(\frac{\bar{\zeta} - \zeta}{z - 1}\right)^k f(z) = \sum_{n \geq 0} r_n \frac{z^n}{n!} \tag{1}$$

and refer to the numbers r_n as the (modified) Taylor expansion coefficients. Being suitably normalized, these numbers become algebraic. More precisely, we have the following.

Proposition 1. *There exists a non-zero complex number ω so that the numbers*

$$b_n = \frac{r_n}{(2\pi i)^n \omega^{k+2n}}$$

are algebraic.

This proposition follows from [9, Main Theorem 1, with $m = 1$] and [10, Theorem 3.2] where a much more general setting is considered. It also follows from the work of Wolfart [12]. In less generality, which is sufficient for our purposes, the proposition is essentially equivalent to the Damerell's theorem [5, Theorem 4.0.4]. The number ω is defined up to an algebraic multiple.

Definition. Let $p > 3$ be a prime. *Call a point ζ in the complex upper half-plane* **suitable** *(with respect to p) if the three points ζ , ζ/p and ζ/p^2 are $SL(2,\mathbb{Z})$-equivalent.*

All suitable points belong to imaginary quadratic fields such that p is a norm of an algebraic integer in that field; if the latter condition is satisfied, the complex upper half-plane contains infinitely many suitable points that lie in this field (see Section 4 for details).

For a suitable point ζ with

$$\frac{a\zeta + b}{c\zeta + d} = \frac{1}{p}\zeta, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

set

$$\pi = c\zeta + d.$$

One can easily check that $\mathbb{Q}(\zeta)$ is an imaginary quadratic field and that π is an algebraic integer in that field that satisfies $\pi\bar{\pi} = p$ (see Section 4).

Theorem 1 *Let ζ be an suitable point, and let $\pi = c\zeta + d$ as above.*

Let

$$\mathcal{E}(n) = 1 - \lambda \frac{p^n}{\pi^{k+2n}} + \frac{p^{k+2n-1}}{\pi^{2k+4n}}.$$

Then there exist a finite extension K of K_0, a place $\mathfrak{p}$ which divides the principal ideal $(\bar{\pi})$ and a choice of $\omega \neq 0$ such that the algebraic numbers $\mathcal{E}(n)b_n$ belong to $\mathcal{O}_{\mathfrak{p}}$, the ring of $\mathfrak{p}$-integers in K and are moments of a measure:

$$\int_{\mathbb{Z}_p^*} x^n d\mu = \mathcal{E}(n) b_n.$$

In particular, these numbers satisfy the following congruences: if

$$n_1 \equiv n_2 \bmod (p-1)p^A$$

with a non-negative integer A, then

$$\mathcal{E}(n_1)b_{n_1} \equiv \mathcal{E}(n_2)b_{n_2} \bmod p^{A+1} \tag{2}$$

Remarks. 1. The congruences (2) are considered as congruences modulo powers of p in the ring $\mathcal{O}_{\mathfrak{p}}$. The numbers involved in (2) are not algebraic integers, but they are $\mathfrak{p}$-integers.

2. In order to make the exposition shorter and clearer we limited ourselves to modular forms of even weight on the full modular group although our methods are flexible enough to allow smooth generalization to higher levels. The natural bound of generality in this direction is that of the Shimura's papers [9, 10], when the algebraicity result (an analog of Proposition 1) is known. Another possible direction to generalize this result was pointed out by B. Mazur: one can introduce the second p-adic variable by considering (rigid-analytic) families of Hecke eigenforms parameterized by the Eigencurve [1] instead of a single modular Hecke eigenform f.

2. Differential operators. For a point $\tau = x + iy$ in the complex upper-half plane we write $q = \exp(2\pi i\tau)$. The Eisenstein series of weight two does not exist; instead, we are going to use two functions: the complex-analytic Ramanujan series

$$P = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$$

and the function

$$S = P - \frac{3}{\pi y}.$$

One obtains S using Hecke's summation trick for the Eisenstein series of weight two. It is well-known that S, though it is not complex-analytic, transforms like a modular form of weight two on the full modular group.

The Ramanujan differential operator

$$D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$$

acts on complex-analytic functions but destroys modularity. We also need the operator

$$\partial_k = \partial = D - \frac{k}{4\pi y}.$$

This operator destroys analyticity, but preserves modularity, namely, it sends modular functions of weight k to modular functions of weight $k+2$. If we apply ∂_k to functions which transform like modular forms of weight k, we will sometimes drop the index k from our notations. Thus ∂^n applied, say, to a modular form f of weight k means $\partial_{k+2n-2}\circ\partial_{k+2n-4}\cdots\circ\partial_k f$.

Proposition 2 *Let F be a function on the upper-half plane which is complex-analytic in a neighborhood of a point α. Write*

$$F(\tau) = (\tau - \bar{\alpha})^{-k} \sum_{n \geq 0} \frac{r_n}{n!} \left(\frac{\tau - \alpha}{\tau - \bar{\alpha}} \right)^n$$

with an integer k. Then

$$r_n = (2\pi i)^{-n} (\alpha - \bar{\alpha})^{k+n} \partial_{k+2n-2} \circ \cdots \circ \partial_{k+2} \circ \partial_k \; F|_{\tau=\alpha}$$

Proof. Set $z = (\tau - \alpha)/(\tau - \bar{\alpha})$. The differentiation ∂_k in terms of the variable z is

$$\partial_k = \frac{1}{2\pi i} \frac{1}{(\alpha - \bar{\alpha})} \left((z-1)^2 \frac{d}{dz} - k \frac{(z-1)(\bar{z}-1)}{z\bar{z}-1} \right).$$

Since $d\bar{z}/dz = 0$ and after applying ∂ repeatedly we are going to set $\tau = \alpha$ and hence $\bar{z} = 0$, we can use the holomorphic differentiation

$$\tilde{\partial}_k = \frac{1}{2\pi i} \frac{1}{(\alpha - \bar{\alpha})} \left((z-1)^2 \frac{d}{dz} - k(z-1) \right)$$

instead of ∂_k. Now the proposition can be easily checked using term-by-term differentiation of the series

$$F(z) = (\bar{\alpha} - \alpha)^{-k} (z-1)^k \sum_{n \geq 0} \frac{r_n}{n!} z^n$$

and induction on n.

3. A special case of the q-expansion principle d'après N. Katz Let $\mathcal{O}$ be the ring of integers in an algebraic number field. For two formal power series $g_1 = \sum b_1(n) q^n$ and $g_2 = \sum b_2(n) q^n$ in $\mathcal{O}[[q]]$ we say $g_1 \equiv g_2 \bmod p^A$ if they are congruent coefficientwise, namely, $b_1(n) \equiv b_2(n) \bmod p^A$ for all n. Let $g = \sum_{(n,p)=1} b(n) q^n$ be the q-expansion of a complex-analytic modular form. The operator D acts on q-expansions in an obvious manner; moreover, $D^{l_1} g \equiv D^{l_2} g \bmod p^{A+1}$ if $l_1 \equiv l_2 \bmod (p-1)p^A$. However, this operator does not preserve the algebraicity of special values. The latter property is preserved by ∂. Thus we want to replace D by ∂ (and P by S), take values at a point ζ, and derive congruences for these values from the congruences for the corresponding q-expansions. The assertion we need is essentially proven in [5]. However, in the absence of a particular statement to refer to in loc. cit. , we sketch the proof here. We closely follow the notations of Katz, and refer the reader to [5] for detailed descriptions of constructions and some definitions.

Since all the modular forms under our consideration are on $\Gamma(p)$, with trivial Nebentypus, according to [5, 5.6], we do not need to consider level structures, and we suppress them from the notations (this corresponds to the case $N_0 = 1$ in the notations of [5]). In particular, we write $R^\bullet(B)$ for the graded ring $R^\bullet(B, \Gamma(p)^{arith})$ of modular forms defined over a ring B, introduced in [5, 2.1.3]. Note that according to [5, 1.1,2.4] homogeneous rational functions in complex-analytic modular forms belong to $R^\bullet(B)$, provided their q-expansion coefficients belong to B. The correspondence, which preserves q-expansions, between elements of $R^\bullet(B)$

and complex-analytic modular forms whose q-expansion coefficients happen to lie in B is described in [5, 2.4.1-2.4.3].

A certain graded algebra of operators $\mathcal{Z}$ acts on C^∞ modular forms [5, 1.6], but does not act stably on the subring $R^\bullet(B)$, for some operators destroy analyticity. For a p-adic ring B (i.e. B is complete and separated in its p-adic topology) [5, 5.0.0] we write $V(B) = V(\mathbb{Z}_p)\hat{\otimes}_{\mathbb{Z}_p}B$ for the p-adic ring $V(B, \Gamma(1))$ which is defined in [5, 5.1.0] and is isomorphic to $V(B, \Gamma(p)^{arith})$ in view of [5, 5.6.14]. One refers to $V(B)$ as the ring of p-adic modular forms. Evaluation on the Tate curve defines an injective q-expansion homomorphism [5, 5.2.1] $V(B) \hookrightarrow \widehat{B((q))}$ =the p-adic completion of $B((q))$. Following [5, 5.9, 5.10], consider the graded subring $GV^\bullet(\mathbb{Z}_p)$ of the non-graded ring $V(\mathbb{Z}_p)$. The action of $\mathcal{Z}$ on elements of $V(\mathbb{Z}_p)$ is described in [5, 5.9]. According to [5, Lemma 5.9.3] the graded ring $GV^\bullet(\mathbb{Z}_p)\otimes_{\mathbb{Z}}\mathbb{Z}[1/12]$ becomes a graded $\mathcal{Z}$-module under this action. In particular, we can attach an even integer $l(\varphi)$, the weight, to any element φ of the ring.

Let K_0 be an imaginary quadratic field, in which p splits, let K be a finite extension of K_0. Fix an embedding $K \hookrightarrow \mathbb{C}$ and consider $\tau \in K_0$ with positive imaginary part. Note that we do not require at this moment τ to be a suitable point. Let $\mathfrak{p}$ be a place of K over p, $\mathcal{O}_\mathfrak{p}$ the ring of $\mathfrak{p}$-integers, $\widehat{\mathcal{O}_\mathfrak{p}}$ its completion and $\widehat{K_\mathfrak{p}}$ the field of fractions of $\widehat{\mathcal{O}_\mathfrak{p}}$. We now have the following inclusions [5, 8.0.4, 8.0.5] of graded rings which respect gradation:

$$R^\bullet(K) \hookrightarrow R^\bullet(\widehat{K_\mathfrak{p}}) = R^\bullet(\mathbb{Z}_p) \otimes \widehat{K_\mathfrak{p}} \hookrightarrow GV^\bullet(\mathbb{Z}_p) \otimes \widehat{K_\mathfrak{p}}, \tag{3}$$

and

$$R^\bullet(K) \hookrightarrow R^\bullet(\mathbb{C}) \hookrightarrow C^\infty(GL^+/\Gamma(p)),$$

where $C^\infty(GL^+/\Gamma(p))$ is the graded ring of (generalized) C^∞-modular forms [5, 1.1]. Elements of $\mathcal{Z}$ act on C^∞-modular forms and preserve algebraicity. Namely, if a complex number Ω is chosen [5, 4.1.8] so that $Q(\tau)/\Omega^4$ and $R(\tau)/\Omega^6$ belong to K, then for $Z \in \mathcal{Z}$ of weight $l(Z)$ and $F \in R^\bullet(K)$ of weight $l(F)$ the number $(ZF)(\tau)/\Omega^{l(Z)+l(F)}$ belongs to K.

The inclusions (3) allow to attach to any element from $ZR^\bullet(K)$, which, in general, does not correspond to a complex analytic function, a formal q-expansion: making use of magic differentials, one defines the map [5, 5.4.7] $R^\bullet(K) \hookrightarrow V(K)$ which preserves q-expansions and is compatible with (3). After that one acts with the elements of $\mathcal{Z}$ on the $GV^\bullet$-side.

Following [5, 4.0.3] we denote by $\mathcal{Z}R^\bullet(K)$ the smallest $\mathcal{Z}$-submodule of $C^\infty(GL^+/\Gamma(p))$ which contains $R^\bullet(K)$; it has the structure of a graded $\mathcal{Z}$-module.

The algebra $\mathcal{Z}$ contains in particular a differential operator of weight 2. This operator is ∂ when acting on the C^∞ - modular forms side, and it is D acting on the p-adic modular forms side (on $GV^\bullet$). Also, the operator of weight 2 "multiplication by S" on the C^∞ - modular forms side corresponds to "multiplication by P" on the p-adic modular forms side (cf. [5, Lemma 5.9.3]).

Lemma 1 *Pick a complex number ω so that $\omega^{p-1} = E_{p-1}(\tau)$.*

Let F and G be to elements of $\mathcal{Z}R^\bullet(K)$ of weights $l(F)$ and $l(G)$ respectively. Denote by $\mathcal{F}$ and $\mathcal{G}$ the corresponding q-expansions.

If $\mathcal{F}, G \in \mathcal{O}_\mathfrak{p}((q))$ and

$$\mathcal{F} \equiv \mathcal{G} \bmod p^A$$

with a positive integer A, then

$$F(\tau)/\omega^{l(F)} \equiv G(\tau)/\omega^{l(G)} \bmod p^A.$$

Proof. Since p is assumed to split in K_0, one can choose $\Omega \in \mathbb{C}^*$ so that the elliptic curve $E_0 = \mathbb{C}/ < \Omega, \Omega\tau >$, which may be a priori defined over K, is defined over $\mathcal{O}_\mathfrak{p}$ and has good ordinary reduction at $\mathfrak{p}$. Since the Hasse invariant of E is 1, the value $E_{p-1}(\tau) \neq 0$ ([4, 2.1]). Therefore $\omega \neq 0$. Moreover, the elliptic curve $E = \mathbb{C}/ < \omega, \omega\tau >$, is defined over $\mathcal{O}_\mathfrak{p}$ and has good ordinary reduction at $\mathfrak{p}$.

Making use of the inclusions (3), consider F and G as elements of $GV^\bullet(\mathbb{Z}_p) \otimes \widehat{\mathcal{O}_\mathfrak{p}}$. As in [5, 2.4], E may be considered as a ($\Gamma(1)$-)test object over $\mathcal{O}_\mathfrak{p}$, and, as explained in [5, 5.10], F and G get values, in $\widehat{\mathcal{O}_\mathfrak{p}}$, by being evaluated at this test object. The Comparison Theorem [5, 8.0.9] implies that these values lie in $\mathcal{O}_\mathfrak{p}$ and their images under the chosen inclusion $K \hookrightarrow \mathbb{C}$ coincide with the complex numbers $F(\tau)/\omega^{l(F)}$ and $G(\tau)/\omega^{l(G)}$. The evaluation homomorphism $GV^\bullet(\mathbb{Z}_p) \to \mathcal{O}_\mathfrak{p}$ is prolongable [5, 5.10.7] to a homomorphism of p-adic rings $GV^\bullet(\mathbb{Z}_p) \otimes \mathcal{O}_\mathfrak{p} \hookrightarrow V(\mathcal{O}_\mathfrak{p}) \to \mathcal{O}_\mathfrak{p}$. Also, the q-expansion principle [5, 5.2] asserts that the q-expansion homomorphism $V(\mathcal{O}_\mathfrak{p}) \hookrightarrow \widehat{\mathcal{O}_\mathfrak{p}((q))}$ is injective and its cokernel $\widehat{\mathcal{O}_\mathfrak{p}((q))}/V(\mathcal{O}_\mathfrak{p})$ is $\mathcal{O}_\mathfrak{p}$-flat. From this we conclude that the images of two elements of $V(\mathcal{O}_\mathfrak{p})$ under the evaluation homomorphism become p-adically close, provided the images of these elements under the q-expansion homomorphism are close. More precisely, this implies the congruences claimed in the lemma.

We are going to make use of the following proposition which follows from Lemma 1.

Proposition 3. *Let $g = \sum_{(n,p)=1} b(n)q^n$ be the q-expansion of a complex-analytic modular form on $\Gamma(p)$ of even weight k and trivial Nebentypus. Assume $b(n) \in \mathcal{O}_\mathfrak{p}$, and pick ω as in Lemma 1. Then there is a $\mathbb{Z}_p^*$ - measure μ such that*

$$\int_{\mathbb{Z}_p^*} x^n d\mu = \omega^{-(k+2n)} \partial^n g(\tau).$$

In particular, the congruence

$$n_1 \equiv n_2 \bmod (p-1)p^A$$

implies the congruence

$$\omega^{-(k+2n_1)} \partial^{n_1} g(\tau) \equiv \omega^{-(k+2n_2)} \partial^{n_2} g(\tau) \bmod p^{A+1}. \tag{4}$$

Proof. Pick $\mathcal{F} = D^{n_1}g$ and $\mathcal{G} = D^{n_2}g$ in Lemma 1, notice that $n_1 \equiv n_2 \bmod (p-1)p^A$ implies that $\mathcal{F} \equiv \mathcal{G} \bmod p^{A+1}$ by the Euler's Theorem, and apply Lemma 1 to obtain (4). More generally, we have the $\mathcal{Z}R^{\bullet}(K)$-valued measure ν with the moments

$$\int_{\mathbb{Z}_p^*} x^n d\nu = D^n g \in \mathcal{O}_{\mathfrak{p}}((q)).$$

Now Lemma 1 allows us to apply the evaluation at τ procedure. Combined with the abstract version of Kummer congruences [6, 4.0.6], this procedure produces the measure μ in question.

4. Suitable points.

Proposition 4.

Let ζ be a suitable with respect to p. Then $\mathbb{Q}(\zeta)$ is an imaginary quadratic field in which p is the norm of an algebraic integer. Moreover, for $p > 3$, any imaginary quadratic field in which p is the norm of an algebraic integer contains infinitely many suitable points.

Proof. Suppose $T(\zeta) = \zeta/p$ for some $T(z) = \dfrac{az+b}{c+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. By taking the imaginary parts of both sides of the above equation, we see that $\dfrac{Im(\zeta)}{|c\zeta+d|^2} = \dfrac{Im(\zeta}{p}$. Therefore $|c\zeta+d|^2 = p$. Also, ζ satisfies $c\zeta^2 + (d-pa)\zeta - pb = 0$. Hence $(c\zeta)^2 + (d-pa)c\zeta - cpb = 0$. This implies that $c\zeta$ is an algebraic integer and so is $\pi = c\zeta + d$. Clearly, $\mathbb{Q}(\zeta)$ is an imaginary quadratic field and π is an algebraic integer in $\mathbb{Q}(\zeta)$ of norm p.

Now suppose K_0 is an imaginary quadratic field and $\pi \in O_{K_0}$ is an algebraic integer of norm p. Then $\zeta = \dfrac{\pi - d}{c}$ is suitable with respect to p if and only if there exist integers a and b such that $(c\zeta)^2 + (d-pa)c\zeta - cpb = 0$. Let $t = \pi + \bar{\pi}$ be the trace of π over $\mathbb{Q}$. Then π is a root of $x^2 - tx + p$ and consequently $c\zeta = \pi - d$ is a root of $x^2 + (2d-t)x + d^2 - td + p$. Consequently, c and d must be chosen so that $p|d-t$ and $cp|d^2 - td + p$. Thus we must take $d \equiv t \bmod p$ and c to be a divisor of $\dfrac{d(d-t)}{p} + 1$ to make sure that $\zeta = \dfrac{\pi - d}{c}$ is $SL(2,\mathbb{Z})$-equivalent to ζ. Now, $\dfrac{\zeta}{p} = \dfrac{\pi - d}{pc}$; therefore, to make ζ suitable we must also make sure that $cp^2|d^2 - td + p$. This is accomplished if we make $\dfrac{d(d-t)}{p} \equiv -1 \bmod p$. For $p > 3$, $t \not\equiv 0 \bmod p$ (this follows from the fact that $t^2 - 4p < 0$). Hence, for $p > 3$, the latter condition becomes $\dfrac{(d-t)}{p} \equiv -t^{-1} \bmod p$, where t^{-1} denotes the inverse of $t \bmod p$, or $d \equiv -t^{-1}p + t \bmod p^2$. Since d can be chosen in infinitely many ways, the number of suitable points ζ in K_0 is infinite.

Note that the linear fractional transformation that maps ζ/p to ζ/p^2 has lower row (cp, d) and consequently $cp\zeta/p + d = \pi$. Hence, a linear fractional transformation $T(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$ that maps ζ to ζ/p^2 satisfies $c_1\zeta + d_1 = \pi^2$. This is the factor of automorphy that we will need in the next section.

5. Action of Hecke operators on the modified Taylor expansion coefficients at a suitable point For a function with a Fourier expansion

$$f(\tau) = \sum_n a(n)q^n, \qquad q = \exp(2\pi i\tau),$$

the function

$$\tilde{f}(\tau) = f(\tau) - \frac{1}{p}\sum_{i=0}^{p-1} f\left(\tau + \frac{i}{p}\right)$$

has the q-expansion

$$\tilde{f} = \sum_{(n,p)=1} a(n)q^n.$$

Under certain circumstances we can express the modified Taylor coefficients of $\tilde{f}$ in terms of Taylor coefficients of f.

Proposition 5.

Let f be a modular form on $SL(2,\mathbb{Z})$ of even weight k which is a Hecke eigenform with an eigenvalue λ for the Hecke operator at p. Suppose ζ to be a suitable point with $\pi = c\zeta + d$ as in the proof of Proposition 4. If

$$f(\tau) = (\tau - \bar{\zeta})^{-k} \sum_{n\geq 0} \frac{r_n}{n!}\left(\frac{\tau - \zeta}{\tau - \bar{\zeta}}\right)^n$$

then

$$\tilde{f}(\tau) = (\tau - \bar{\zeta}/p^2)^{-k} \sum_{n\geq 0} \left(p^{-2n-2k}\pi^{2k+4n} - \lambda p^{-n-2k}\pi^{k+2n} + p^{-k-1}\right) \frac{r_n}{n!}\left(\frac{\tau - \zeta/p^2}{\tau - \bar{\zeta}/p^2}\right)^n$$

Proof. Representing the Hecke operator T_p as a matrix operator and taking into the account that f is its eigenform with eigenvalue λ, we obtain

$$\lambda f(\tau) = \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{\tau + i}{p}\right) + p^{k-1} f(p\tau).$$

Thus

$$\tilde{f}(\tau) = f(\tau) - \lambda f(p\tau) + p^{k-1} f(p^2\tau). \tag{5}$$

Write $r_n = r_n(\alpha)$ for the modified Taylor expansion coefficients of f to indicate the dependence on the imaginary quadratic point α in the upper-half plane:

$$f(\tau) = (\tau - \bar{\alpha})^{-k} \sum_{n\geq 0} \frac{r_n(\alpha)}{n!} \left(\frac{\tau - \alpha}{\tau - \bar{\alpha}} \right)^n .$$

Making use of (5), we obtain the expansion of $\tilde{f}$ at $\alpha = \zeta/p^2$:

$$\tilde{f}(\tau) = (\tau - \bar{\zeta}/p^2)^{-k} \sum_{n\geq 0} \frac{1}{n!} \left(r_n(\zeta/p^2) - \lambda p^{-k} r_n(\zeta/p) + p^{-k-1} r_n(\zeta) \right) \left(\frac{\tau - \zeta/p^2}{\tau - \bar{\zeta}/p^2} \right)^n . \tag{6}$$

The operator ∂ preserves modularity. According to Proposition 2, the numbers $(2\pi i)^n(\alpha - \bar{\alpha})^{k+n} r_n(\alpha)$ appear as values at $\tau = \alpha$ of (not complex-analytic for $n > 0$) modular forms. Making use of the factor of automorphy, computed after the Proposition 4, for a suitable point ζ, we obtain the relations

$$r_n(\zeta/p) = p^{-n-k} \pi^{k+2n} r_n(\zeta); \qquad r_n(\zeta/p^2) = p^{-2n-2k} \pi^{2k+4n} r_n(\zeta). \tag{7}$$

The claim of Proposition 5 follows from (7) and (6).

6. Conclusion of the proof of Theorem 1. To prove Theorem 1 notice that the function $\tilde{f}$ from previous section satisfies the hypothesis of Proposition 3, and apply the conclusion of Proposition 3 to the modified Taylor expansion of this function obtained in Proposition 5. This is the modified Taylor expansion at the point ζ/p^2, and therefore, when applying Proposition 3, we have to use the complex number ω corresponding to this point. Since ζ is assumed to be suitable we conclude from Proposition 4b that $E_{p-1}(\zeta/p^2) = \pi^{2(p-1)} E_{p-1}(\zeta)$. It now follows from the choice of ω in Proposition 3 that it is sufficient to divide the number ω corresponding to ζ by π^2, and to make use of the connection between the powers of ∂ and the modified Taylor expansion coefficients established in Proposition 2 with $\alpha = \zeta/p^2$.

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