JACOBI FORMS AND A TWO-VARIABLE p-ADIC L-FUNCTION

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Introduction. Consider a Jacobi form $\phi(\tau, z) = \sum_{n,r} c(n,r) q^n \zeta^r$ whose Fourier coefficients $c(n,r)$ are algebraic numbers. Let p be an odd prime. In this paper we associate to ϕ a Λ-adic p-ordinary form in the sense of [4]. The construction comes from the map D_ν introduced in [2], Theorem 3.1. This map associates to a Jacobi form a family of modular forms parametrised by ν. We obtain the two-variable p-adic interpolation of special values of symmetric squares of elliptic cusp forms as an application of our construction. This result is closely connected to [6], Theorem I. However, our approach is rather different and much more explicit. Our methods of p-adic interpolation are based on the methods and results of A.A.Panchishkin [10], [11], [12], [13]. We use the technique of Jacobi forms instead of the technique of modular forms of half integral weight used in [13], [6]. It yields considerable simplifications. In particular, we do not need nor the calculation of the trace of certain modular forms of higher level, nor the technique of non-holomorphic modular forms. Our methods might be generalised to the case of Siegel modular forms. The one-variable p-adic interpolation of the special values of the standard L-function (namely, the interpolation along the cyclotomic line) in this case is constructed by A.A.Panchishkin [10]. Recently S.Böcherer and C.-G.Schmidt [1] obtained a result of such type using a different method in a more general setting. The paper of J.Tilouine and E.Urban [15] extends the H.Hida's theory of ordinary Λ-adic forms to the case of Siegel modular forms. One might expect that our construction is applicable to this case and would yield the two-variable interpolation of special values of the standard L-functions associated with Siegel modular forms.

The content of the paper is as follows. In the first section we construct a Λ-adic p-ordinary modular form associated to a Jacobi form (theorem 1 in the text). Some basic facts from p-adic analysis are also recalled in this section. The main references here are [10] and [4]. The Jacobi - Eisenstein series $E_M^{\xi,\psi}$ is introduced in the second section. This

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section is devoted to the computation of its Fourier expansion coefficients (theorem 2 in the text). We apply the construction of section 1 to this Jacobi - Eisenstein series, and get a two-variable p-adic family of modular forms in the third section. In section 4 we use the classical Rankin's method in order to connect the modular forms from our two-variable family with special values of symmetric squares. The results of sections 3 and 4 allow to construct the two-variable p-adic interpolation of special values of the symmetric square of a p-ordinary Λ-adic Hecke eigenform (section 5).

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1. A Λ-adic form associated with a Jacobi form. We refer to [10], Chapter I for the basic definitions and detailed explanations concerning to the p-adic distributions and p-adic measures.

In particular, in order to establish existence of a measure we use the following very useful criterion:

Proposition 1. *([10], Chapter I, Proposition 3.3)*

Let $\{\varphi_i\}$ be a family of continuous functions from $\mathbb{Z}_p^*$ to the ring of integers $\mathcal{O}_p$ in the Tate's field $\mathbb{C}_p$. Assume that the set of finite $\mathbb{C}_p$-linear combinations of $\{\varphi_i\}$ is dense in the space of all such functions. Let $\{a_i\}$ be a family of elements in $\mathcal{O}_p$. Then the existence of a measure μ with the property

$$\int_{\mathbb{Z}_p^*} \varphi_i \mu = a_i$$

is equivalent to the following statement.

For every finite set of elements $b_j \in \mathbb{C}_p$ it follows from

$$\{\sum_j b_j f_j(y) \in p^n \mathcal{O}_p \text{ for every } y \in \mathbb{Z}_p^*\}$$

that

$$\{\sum_j b_j a_j \in p^n \mathcal{O}_p\}$$

.

This proposition is called the abstract Kummer congruences. As its first application we recall

Proposition 2. *Let χ mod D be a fixed primitive Dirichlet character, $(p, D) = 1$. For an integer $c > 1$ such that $(c, pD) = 1$ there exists a* *measure* $\mu(c,\chi)$ *such that*

$$\int_{\mathbb{Z}_p^*} \xi x^s \mu(c,\chi) = (1-\bar{\xi}\chi(c)c^{-s})L(1-s,\xi\bar{\chi})$$

for a positive integer s *and a primitive nontrivial Dirichlet character* $\xi \bmod p^\beta$.

The L-function in the right hand side of the assertion is the Dirichlet L-function. Its values at non-positive integers are known to be algebraic numbers.

One reduces the proof to the verification of the criterion given in proposition 1 and the standard properties of Mazur's p-adic Mellin transform (see [11], p. 151, [10], 4.5 section 3).

In this paper we deal not only with $\mathbb{C}_p$ (actually $\mathcal{O}_p$) - valued measures, but also with measures taking values in the ring of formal power series $\mathcal{O}_p[[q]]$. We consider this ring as complete $\mathcal{O}_p$-module endowed with the coefficient-wise topology. Another important tool is the theorem ([10], Chapter I, Theorem 4.5) which establishes the one-to-one correspondence (which is in fact an isomorphism of the algebras) between the p-adic measures and bounded p-adic analytic functions on $T = \{z \in \mathbb{C}_p^*, |t-1|_p < 1\}$. These can be represented as power series from $\mathcal{O}_p[[X]]$. Given a measure μ, one gets the p-adic analytic function associated with it as the p-adic Mellin transform

$$L_\mu(x) = \int_{\mathbb{Z}_p^*} x\mu.$$

Here x belongs to the group $Hom_{\mathbf{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$. This is a p-adic analytic Lie group which is isomorphic to several copies of T. Note that each element of the torsion subgroup of this group can be identified with a Dirichlet character modulo a power of p. (We assume that the embedding $\bar{\mathbb{Q}} \to \mathbb{C}_p$ is fixed once and for all, and identify algebraic numbers with their images in $\mathbb{C}_p$.) Each integer $l \in \mathbb{Z}$ can also be considered as an element of this group which sends $y \in \mathbb{Z}_p^*$ into y^l. In the following we will often consider pairs (l,χ) of an integer l and a Dirichlet character χ modulo a power of p as elements of $Hom_{\mathbf{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$. We write $\int x^l \chi\mu$ for $\int z\mu$ if $z = (l,\chi)$.

The above consideration allows to reformulate the definition of Λ-adic forms ([4], equivalent definitions (Λ) on p. 196, (Λ') on p. 208). Let ϑ be a non-trivial Dirichlet character modulo a positive power of p.

Definition 1. *A measure μ on $\mathbb{Z}_p^*$ with values in $\mathcal{O}_p[[q]]$ is called a Λ-adic form of character ϑ if*

$$\int_{\mathbb{Z}_p^*} x^k \chi\mu \in M_{M+k}(\Gamma_0(p^z), \chi\vartheta\omega^{-k}).$$

for integers M and $z > 0$ and for almost all positive integers k. Here ω denotes the Teichmüller character.

The symbol $M_k(\Gamma_0(N), \chi)$ denotes as usual the space of (complex-analytic) modular forms of level N, weight k and of character χ. When the above integral is a p-ordinary form for all sufficiently large k then μ is called a p-ordinary Λ-adic form. The equivalence of our definition and the original one follows from [4], Theorem 7.3.2, p.214.

Let ϕ be a Jacobi form of weight M, index m on congruence subgroup $\Gamma_0(N)$ of a Dirichlet character ξ modulo N. In accordance with the basic definitions (see [2]) it means that ϕ is a function in two variables with the Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{n,r \\ r^2 \leq 4nm}} c(n,r) q^n \zeta^r \qquad q = \exp(2\pi i\tau), \qquad \zeta = \exp(2\pi i z).$$

It satisfies the following transformation properties. For every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

$$\phi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^M \xi(d) \exp\left(2\pi i \frac{cmz^2}{c\tau+d}\right) \phi(\tau, z).$$

For a couple $(\lambda, \mu) \in \mathbb{Z}^2$

$$\phi(\tau, z + \lambda\tau + \mu) = \exp(-2\pi i m(\lambda^2\tau + 2\lambda z)\phi(\tau, z).$$

We assume that $c(n,r)$ are algebraic integers. Denote $\tau(\chi)$ the Gauss sum associated with a Dirichlet character χ. One should not confuse it with the complex variable τ. As usual, U_p denotes the operator which sends $\sum a(n)q^n$ to $\sum a(pn)q^n$.

Theorem 1. *Let $y \in Hom_{\mathbf{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$. Suppose $N = p^\beta$. Put*

$$a(n, y) = \lim_{l\to\infty} \sum_{\substack{n,r \\ r^2 \leq 4p^{l!}nm}} y(r) c(p^{l!}n, r).$$

Then the p-adic limit in the right-hand side exists and the numbers $a(n, y)$ are Fourier coefficients of a Λ-adic p-ordinary modular form of character ξ.

Remark. In accordance with the definition 1 the theorem asserts that there exists a $\mathcal{O}_p[[q]]$-valued measure μ_ϕ associated with a Jacobi form ϕ such that

$$\int_{\mathbb{Z}_p^*} x^\nu \chi \mu_\phi = \sum_{n\geq 0} q^n \lim_{l\to\infty} \sum_{r^2\leq 4p^{l!}nm} r^\nu \chi(r) c(p^{l!}n, r) \in M^{\mathbf{ord}}_{M+\nu}(\Gamma_0(p^x), \chi\omega^{-\nu}\xi).$$

for some positive x. In other words, the "moments" of this measure are p-ordinary modular forms. Here ν is a non-negative integer, and χ is a Dirichlet character modulo a power of p.

Proof. Let χ be a Dirichlet character modulo N_1. Consider the function $\phi_\chi(\tau, z) = \sum \chi(r)c(n,r)q^n\zeta^r$. We claim that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_2)$

$$\phi_\chi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^M \chi\xi(d) \exp\left(2\pi i \frac{cmz^2}{c\tau+d}\right)\phi_\chi(\tau, z),$$

where $N_2 = l.c.m.(N, N_1^2)$. In order to prove this claim recall ([2], Theorem 1.4) the group G^J which consists of triples $[\gamma, X, \sigma]$, $(\gamma \in SL_2(\mathbb{R}), X \in \mathbb{R}^2, \sigma \in \mathbb{C},\ |\sigma| = 1)$. The group low is given by $[\gamma, X, \sigma][\gamma', X', \sigma'] = [\gamma\gamma', X\gamma' + X', \sigma\sigma' \exp\left(2\pi i \det \binom{X\gamma'}{X'}\right)]$. This group acts on the functions in two variables via

$$\phi|_{M,m}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda,\mu), \sigma\right](\tau, z) = \sigma^m (c\tau+d)^{-M}$$

$$\times \exp\left(2\pi i m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

One has

$$\tau(\chi)\phi_\chi = \sum_{u \bmod N_1} \bar{\chi}(u)\phi|_{M,m}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \left(0, \frac{u}{N}\right), 1\right].$$

It yields that

$$\tau(\chi)\phi_\chi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)(c\tau+d)^{-M}\exp\left(2\pi i\frac{cmz^2}{c\tau+d}\right) =$$

$$\sum_{u \bmod N_1} \bar{\chi}(u)\phi|_{M,m}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\frac{cu}{N}, 0), 1\right]\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (0, \frac{du}{N}), 1\right](\tau, z) =$$

$$\xi(d) \sum_{u \bmod N_1} \bar{\chi}(u)\phi|_{M,m}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (0, \frac{du}{N}), 1\right](\tau, z)$$

$$\xi\chi(d)\tau(\chi)\phi_\chi(\tau, z),$$

and our claim is proved.

It now follows from [2], theorem 3.2 (and its proof) that

$$D_\nu(\phi) = \sum_{n\geq 0} q^n \sum_{r^2\leq 4nm} \sum_{0\leq\mu\leq\nu} \frac{(M+\nu-\mu-2)!}{(M+\nu-2)!} \frac{(-nm)^\mu r^{\nu-2\mu}}{\mu!(M-2\mu)!} \chi(r)c(n,r)$$

is a modular form of weight $M+\nu$ of character $\chi\xi$ on congruence subgroup $\Gamma_0(N_2)$. (This modular form differs from those defined in [2], (8), p.32 by the factor $(2\pi i)^\nu$. There it was denoted by ξ_ν.) Applying the ordinary projector $\mathcal{E} = \lim_{l\to\infty} U_p^{l!}$ to this modular form we obtain the p-ordinary modular form

$$\mathcal{E}D_\nu(\phi) = \sum_{n\geq 0} q^n \lim_{l\to\infty} \sum_{r^2\leq 4p^{l!}nm} r^\nu \chi(r) c(p^{l!}n, r).$$

The p-adic limit in the right-hand side exists since it exists in the left. The application of the abstract Kummer congruences finishes the proof.

Remark. We should mention a relevant construction of A.Sofer. She (a lecture given in Max-Plank-Institut, Bonn) considered the Taylor expansion coefficients of a Jacobi form as moments of a p-adic measure. These are not holomorphic modular forms, but p-adic modular forms in the sense of N.M.Katz [8].

Theorem 1 says that the modular form $\mathcal{E}D_\nu(\phi)$ depends analytically on the index ν. Consider a p-adic analytic family of Jacobi forms ϕ_M. We will not give here a precise definition of such family. We just assume that the Fourier expansion coefficients of such forms depend analytically on M after a suitable regularization procedure. Then the Fourier coefficients of the modular forms $\mathcal{E}D_\nu(\phi)$ should depend analytically on both variables ν and M. In the next section we take certain Jacobi - Eisenstein series as ϕ_M and we show that this is actually the case. Of course, we consider the integers ν and M as their images in $Hom_{\mathbf{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ under the natural embedding.

2. The Jacobi - Eisenstein series $E_M^{\xi,\psi}$ and its Fourier expansion. Let $M\geq 3$ be an integer. Let ξ and ψ be primitive Dirichlet characters modulo p^β and p^x respectively. Put $\alpha = \max(\beta, x)$. From now on we introduce the condition *The conductor of the character $\xi\psi$ equals p^α; both $\beta, x > 0$.* We will always suppose that the characters ξ and ψ satisfy this condition. It is not too restrictive, and yields considerable simplifications of the calculations.

Proposition 3. *The series*

$$E_M^{\xi,\psi} = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c\equiv 0 \bmod p^\beta}} \bar{\xi}(d) \times \sum_{\lambda\in\mathbb{Z}} \psi\xi(\lambda)(c\tau+d)^{-M} \exp\left(2\pi i\left(\lambda^2\frac{a\tau+b}{c\tau+d} + 2p^\alpha\lambda\frac{z}{c\tau+d} - p^{2\alpha}\frac{cz^2}{c\tau+d}\right)\right)$$

is a Jacobi form of weight M, index $p^{2\alpha}$ with respect to $\Gamma_0(p^\beta)$ with character ξ.

The proof reduces to the direct check and we omit it. The purpose of this section is to compute the Fourier expansion coefficients of $E_M^{\xi,\psi}$.

Theorem 2. *Let*

$$E_M^{\xi,\psi} = \sum_{\substack{n,r \\ r^2\le 4p^{2\alpha}n}} c(n,r)q^n\zeta^r.$$

For a couple (n,r) such that $r^2 < 4p^{2\alpha}n$ we put $r^2-4p^{2\alpha}n = D_0C^2$. Here D_0 is a fundamental discriminant and C is a positive integer. We denote by χ the quadratic character associated with $\mathbb{Q}(\sqrt{r^2-4p^{2\alpha}n})$.

If $n\equiv 0 \bmod p^\alpha$ then

$$c(n,r) = 2^{2M-3}\pi^{2M-2}p^{\alpha(3-2M)+(x+\beta)(1-M)}\Gamma(2M-2)^{-1}\frac{\tau(\xi)\tau(\psi)}{\tau(\xi\bar{\psi})} \times\xi\psi\left(\frac{r}{p^\beta}\right)\frac{L(2-M,\bar{\psi}\chi)}{L(2M-2,\psi^2)}\sum_{d|C}\mu(d)\chi(d)\bar{\psi}(d)d^{M-2}\sum_{l|C/d}l^{2M-3}\bar{\psi}^2(l).$$

Here μ denotes the Möbius function. We understand that $\xi\psi\left(\frac{r}{p^\beta}\right)=0$ if the number r/p^β is not an integer.

Proof. The proof runs along the same lines as those of [2], Theorem 2.1.

First, we split the sum into two parts, according as $c=0$ or not. The first part vanishes. After that we use the identity

$$\lambda^2\frac{a\tau+b}{c\tau+d} + 2p^\alpha\lambda\frac{z}{c\tau+d} - p^{2\alpha}\frac{cz^2}{c\tau+d} = -\frac{c(p^\alpha z-\lambda/c)^2}{c\tau+d} + \frac{a\lambda^2}{c}$$

and the calculation of the Fourier expansion coefficients of the double periodic function

$$F_{k,m}(\tau,z) = \sum_{l,q\in\mathbb{Z}}(\tau+l)^{-M}\exp\left(2\pi i m\left(-\frac{(z+q)^2}{\tau+p}\right)\right).$$

We obtain

$$E_M^{\xi,\psi}(\tau,z) = p^{-M\beta} \sum_{\substack{n,r \\ r^2 \le 4p^{2\alpha}n}} \gamma(n,r) \sum_{c\ge 1} c^{-M} S(c) q^n \zeta^r.$$

Here

$$\gamma(n,r) = i^M \frac{\pi^{M-1/2}}{2^{M-2}\Gamma(M-1/2)} p^{2\alpha(1-M)} (4p^{2\alpha}n - r^2)^{M-3/2},$$

and

$$S(c) = \sum_{\substack{d \bmod p^\beta c \\ (d,pc)=1}} \sum_{\lambda \bmod p^{\alpha+\beta}c} \bar{\xi}(d)\psi\xi(\lambda) \exp\left(2\pi i \left(\frac{a\lambda^2}{p^\beta c} - r\frac{\lambda}{p^{\alpha+\beta}c} + n\frac{d}{p^\beta c}\right)\right).$$

We denote $Q(\lambda) = p^\alpha\lambda^2 - r\lambda + p^\alpha n$ and transform the latter sum to the form

$$S(c) = p^{-\alpha} \sum_{\lambda \bmod p^{\alpha+\beta}c} \psi\xi(\lambda) \sum_{\substack{d \bmod p^{\alpha+\beta}c \\ (d,pc)=1}} \psi(d) \exp\left(2\pi i \frac{dQ(\lambda)}{p^{\alpha+\beta}c}\right).$$

Now put $c = p^\gamma c_0$ and assume that $(p, c_0) = 1$ and $\gamma \ge 0$. Using the well-known lemma ([14], Lemma 3, p.87) we get

$$S(c) = p^{\beta+\gamma-x} c_0 \tau(\psi)\psi(c_0) \sum_{l|c_0} \mu\left(\frac{c_0}{l}\right) N_{l_0}(r^2 - 4p^{2\alpha}n)$$
$$\times \sum_{\substack{\lambda \bmod p^{\alpha+\beta+\gamma} \\ Q(\lambda)\equiv 0 \bmod (p^{\alpha+\beta+\gamma-x})}} \psi\xi(\lambda)\bar{\psi}\left(\frac{Q(\lambda)}{p^{\alpha+\beta+\gamma-x}}\right).$$

Here we used the notation from [2] for the number of solutions of the congruence

$$N_l(r^2 - 4p^{2\alpha}n) = \#\{\lambda \bmod l | Q(\lambda) \equiv 0 \bmod l\}$$

Summarising, we have obtained

$$c(n,r) = \gamma(n,r) p^{\beta(1-M)-x} \tau(\psi)$$
$$\times \sum_{\substack{c_0 \ge 1 \\ (p,c_0)=1}} c_0^{1-M} \psi(c_0) \sum_{l|c_0} \mu\left(\frac{c_0}{l}\right) N_l(r^2 - 4p^{2\alpha}n)$$
$$\times \sum_{\gamma\ge 0} p^{\gamma(1-M)} \sum_{\substack{\lambda \bmod p^{\alpha+\beta+\gamma} \\ Q(\lambda)\equiv 0 \bmod (p^{\alpha+\beta+\gamma-x})}} \psi\xi(\lambda)\bar{\psi}\left(\frac{Q(\lambda)}{p^{\alpha+\beta+\gamma-x}}\right).$$

The first sum, namely the sum over c_0 has an Euler product. It allows to calculate it in terms of special values of certain Dirichlet L-series. (See [7], Proposition 2, p. 69, where a similar calculation is discussed in details.)

Consider the second sum, namely the sum over γ. One can show that the sum over λ is zero as soon as $\gamma \geq x + \alpha - \beta$. It means that the sum over γ is in fact finite and we have already computed all the Fourier coefficients $c(n,r)$ in some closed form. However, we will be able to analyse their properties only after involving the additional condition $n \equiv 0 \bmod p^\alpha$. The following proposition asserts that under this condition all the terms in the sum over γ but this with $\gamma = 0$ vanish. The only non-zero term becomes essentially a Jacobi sum and we compute it in terms of Gauss sums.

Proposition 4. *If $n \equiv 0 \bmod p^\alpha$ then*

$$\sum_{\gamma \geq 0} p^{\gamma(1-M)} \sum_{\substack{\lambda \bmod p^{\alpha+\beta+\gamma} \\ Q(\lambda)\equiv 0 \bmod (p^{\alpha+\beta+\gamma-x})}} \psi\xi(\lambda)\bar{\psi}\left(\frac{Q(\lambda)}{p^{\alpha+\beta+\gamma-x}}\right)$$
$$= p^\alpha \xi\bar{\psi}\left(\frac{r}{p^\beta}\right)\psi(-1)\frac{\tau(\xi)\tau(\bar{\psi})}{\tau(\xi\bar{\psi})}.$$

Proof. Denote for convenience

$$\sum_\lambda = \sum_{\substack{\lambda \bmod p^{\alpha+\beta+\gamma} \\ Q(\lambda)\equiv 0 \bmod (p^{\alpha+\beta+\gamma-x})}} \psi\xi(\lambda)\bar{\psi}\left(\frac{Q(\lambda)}{p^{\alpha+\beta+\gamma-x}}\right)$$
$$= \sum_{\substack{\lambda \bmod p^{\alpha+\beta+\gamma} \\ p^\alpha\lambda\equiv r \bmod (p^{\alpha+\beta+\gamma-x})}} \psi\xi(\lambda)\bar{\psi}\left(\frac{p^\alpha\lambda - r}{p^{\alpha+\beta+\gamma-x}}\right).$$

Consider the case when $x \geq \beta + \gamma$. In this case $p^\alpha\lambda \equiv r \bmod p^{\alpha+\beta+\gamma-x}$ is equivalent to $r \equiv 0 \bmod p^{\alpha+\beta+\gamma-x}$. It yields

$$\sum_\lambda = \sum_{\lambda \bmod p^{\alpha+\beta+\gamma}} \xi(\lambda)\bar{\psi}\left(p^{x-\beta-\gamma}\lambda - \frac{r}{p^{\alpha+\beta+\gamma-x}}\right)$$

Put $\lambda = \lambda_0 + p^\beta t$ for λ_0 running modulo p^β and t modulo $p^{\alpha+\gamma}$. One has

$$\sum_\lambda = \sum_{\substack{\lambda_0 \bmod p^\beta \\ t \bmod p^{\alpha+\beta}}} \xi(\lambda_0)\bar{\psi}\left(p^{x-\beta-\gamma}\lambda_0 + p^{x-\gamma}t - \frac{r}{p^{\alpha+\beta+\gamma-x}}\right).$$

If $p|\left(p^{x-\beta-\gamma}\lambda_0 - \frac{r}{p^{\alpha+\beta+\gamma-x}}\right)$ then it is evident that $\sum_\lambda = 0$. Otherwise there exists y which depends on λ_0 such that $y\left(p^{x-\beta-\gamma}\lambda_0 - \frac{r}{p^{\alpha+\beta+\gamma-x}}\right) \equiv$

1 mod p^x. It follows that

$$\sum_{\lambda} = \sum_{\lambda_0 \bmod p^\beta} \xi(\lambda_0)\bar{\psi}\left(p^{x-\beta-\gamma}\lambda_0 - \frac{r}{p^{\alpha+\beta+\gamma-x}}\right) \sum_{t \bmod p^{\alpha+\gamma}} \bar{\psi}(1+p^{x-\gamma}ty),$$

and the inner sum is zero as soon as $\gamma \neq 0$. If $\gamma = 0$ then the inner sum equals to p^α. If $x > \beta$ then

$$\sum_{\lambda} = p^\alpha \sum_{\lambda \bmod p^\beta} \xi(\lambda)\bar{\psi}\left(p^{x-\beta}\lambda - \frac{r}{p^{\alpha+\beta-x}}\right) = p^\alpha \xi\bar{\psi}\left(\frac{r}{p^{\alpha+\beta-x}}\right) \sum_{\lambda \bmod p^\beta} \xi(\lambda)\bar{\psi}(\lambda p^{x-\beta}-1).$$

The last sum is a Jacobi sum and we need the following lemma in order to express it in terms of Gauss sums.

Lemma 1. *([9], Chapter I, 2.2., [16], Chapter 6.1.) Let χ_1 and χ_2 be primitive Dirichlet characters modulo p^a and p^b respectively. Suppose that $a, b > 0$. Then*

$$\sum_{y \bmod p^a} \chi_1(y)\chi_2(1 - yp^{b-a}) = \frac{\tau(\chi_1)\tau(\chi_2)}{\tau(\chi_1\chi_2)}.$$

Application of this lemma finishes the proof of the proposition in the case when $x > \beta$. The cases $x < \beta$ and $x = \beta$ are considered after the same fashion. The consideration of the latter case involves the condition that p^α is precisely the conductor of $\xi\psi$. Theorem 2 is proved.

3. The two-variable p-adic family of modular forms. Let us now apply the construction of our theorem 1 to the (suitably normalised) Jacobi form $E_M^{\xi,\psi}$. There are two reasons not to apply theorem 1 directly. Firstly, this theorem gives a one-variable family of modular forms, but we actually have two p-adic variables. Namely, we have to investigate the dependence of the p-ordinary modular forms constructed in this way both on (ξ, ν) and on (ψ, M). The second reason is that we have already obtained enough information about the Fourier coefficients $c(n, r)$ of our Jacobi - Eisenstein series in order to construct the desired measure directly.

Proposition 5. *Let $c > 1$ be an integer, $(p, c) = 1$. Put*

$$A = A(M, \nu, \xi, \psi) = 2^{-2M+3}\pi^{-2M+2}(1 - \psi^2(c)c^{2-2M})p^{\alpha(2M-3)+(M-1)(\beta+x)+\beta\nu}$$
$$\times\Gamma(2M-2)\frac{\tau(\xi\bar{\psi})}{\tau(\xi)\tau(\psi)}L(2M-2, \psi^2)$$

There exists a unique two-variable measure μ^c on $\mathbb{Z}^* \times \mathbb{Z}^*$ with values in $\mathcal{O}_p[[q]]$ such that

$$\int_{\mathbb{Z}^*\times\mathbb{Z}^*} \psi\xi(x)x^\nu\bar{\psi}(y)y^{M-1}\mu^c = AU_p^{2(\alpha-\beta)}\mathcal{E}D_\nu(E_M^{\xi,\psi}).$$

Remark. It is evident that the moments of this measure are in fact p-ordinary modular forms. Fix a (complex-analytic) p-ordinary Hecke eigenform f of weight k and of character ξ. Put $M+\nu=k$. After the change of variable $y=x^{-1}$ we get a one-variable measure. The moments of this measure are p-ordinary modular forms of the same weight k and character ξ. They belong to a fixed finite-dimensional space, and one can consider the linear functional presented by the Petersson scalar inner product with f. This construction is quite similar to those of [13], [3]. It yields the one-variable p-adic interpolation of the special values of symmetric square of f.

Proof. Consider the Jacobi form $\phi(\tau,z)=Ap^{-\beta\nu}E_M^{\xi,\psi}(\tau,z)$.

Combining the construction of theorem 1 with the formula for the Fourier coefficients of theorem 2, we obtain:

$$\mathcal{E}D_\nu(\phi)=\sum_{n\geq 0}q^n\lim_{l\to\infty}\sum_{r^2\leq 4p^{l!+2\alpha}n}r^\nu\xi\psi\left(\frac{r}{p^\beta}\right)(1-\psi^2(c)c^{2-2M})L(2-M,\bar{\psi}\chi)$$

$$\times\sum_{d|C}\mu(d)\chi(d)\bar{\psi}(d)d^{M-2}\sum_{l|C/d}\bar{\psi}^2(l)(l)^{2M-3},$$

where $D=D_0C^2=r^2-4p^{l!+2\alpha}n$, and the character χ is associated to $\mathbb{Q}(\sqrt{D})$. The p-adic limit in the right-hand side exists because it exists in the left-hand side. We change the variable $r\to p^\beta r$ and apply the operator $U_p^{2\alpha-2\beta}$ to both sides of the above equality. Notice that $1-\psi^2(c)c^{2-2M}=(1-\chi\psi(c)c^{1-M})(1+\chi\psi(c)c^{1-M})$. After that we apply proposition 2. One might encounter the situation when $r^2-4p^{l!+2\alpha}$ is not coprime to c. In this case it is sufficient to pick another value c_1 which is congruent to the previous one modulo a (sufficiently large) power of p. This consideration allows to finish the proof using the abstract Kummer congruences (proposition 1) coefficient-wise.

4. Reinterpretation in terms of the special values of symmetric square. The p-ordinary modular forms constructed in proposition 5 as moments of the two-variable measure are connected with the special values of symmetric square.

Let f be a (complex-analytic) cusp form of weight k on congruence subgroup $\Gamma_0(p^\beta)$ with character ξ. We suppose that f is a Hecke eigenform. Let $\alpha(q)$ and $\beta(q)$ denote the roots of Hecke polynomial. The symmetric square of the cusp form f twisted with a Dirichlet character ϑ is defined by

$$\mathcal{D}(s,f,\vartheta)=\prod_q(1-\vartheta(q)\alpha(q)^2q^{-s})^{-1}(1-\vartheta(q)\alpha(q)\beta(q)q^{-s})^{-1}(1-\vartheta(q)\beta(q)^2q^{-s})^{-1}.$$

The product converges and the quantities $\pi^{k-2m-1} < f,f >^{-1} D(m,f,\vartheta)$ are known to be algebraic for integer m when $k < m \leq 2k-2$. Here and in the following $< \bullet, \bullet >$ denotes the usual Petersson scalar product.

Proposition 6. *Let $M \geq 3$ and $\nu \geq 0$ be integers. Let the weight of the cusp form f be equal $M+\nu$. The assumptions concerning the Dirichlet characters ξ and ψ are as above. Put*

$$B = p^{\alpha\nu}2^{-2M-2\nu+2}\pi^{-M+1-\nu}\frac{\Gamma(2M+\nu-2)\Gamma(M-1)}{\Gamma(\nu+1)\Gamma(2M-2)}$$

One has

$$< f, \xi_\nu(E_M^{\xi,\psi}) >= BL(2M-2,\bar{\psi}^2)^{-1}\mathcal{D}(2M+\nu-2,f,\bar{\psi}\bar{\xi}) < f,f >^{-1}$$

As it was pointed out to the author by ProfessorBöcherer, one can prove this proposition using Rankin's method directly. Such proof is completely similar to the proof of Proposition 6 in [17]. Another way to prove this proposition is to reduce it to Proposition 6 in [17]. This way was discussed in details in [3], main theorem. It is based on the transformation properties of the twisted theta-function $\theta_\psi(\tau,z) = 1/2\sum_{\lambda\in\mathbb{Z}}\xi\psi(\lambda)\exp(2\pi i(\lambda^2\tau+2\lambda p^\alpha z))$.

5. Two-variable p-adic interpolation of the symmetric square. Let u be the topological generator of $\mathbb{Z}_p^*$, say $u = 1+p$. Let F be a Λ-adic p-ordinary form. We now return to the original definition ([4], Definition (Λ), p.196) and consider F as a formal power series in $\mathcal{O}_p[[Z]][[q]]$. Substituting $Z = \xi(u)u^k - 1$ one obtains the q-expansion of a p-ordinary holomorphic cusp form $F(\xi,k)$ of weight k. Denote by $\Omega(\xi,k) =< F(\xi,k), F(\xi,k) >$ its Petersson scalar square. Denote by $a_n(\xi,k)$ the value which takes the coefficient of q^n of $F(\xi,k)$.We assume $a_1(\xi,k) = 1$ (see [4], Theorem 6.3.7).

For $c > 1$ coprime to p put

$$Q = 2^{-4M-2\nu+6}\pi^{-3M-\nu+3}p^{\alpha(2M+\nu-3)+(M-1)(\beta+x)-\beta\nu}$$

$$\times\frac{\Gamma(2M+\nu-2)\Gamma(M-1)}{\Gamma(\nu+1)}\frac{\tau(\bar{\xi}\psi)}{\tau(\bar{\xi})\tau(\bar{\psi})}(1-\psi^2(c)c^{2-2M}).$$

Theorem 3. *Denote by K the field of fractions of $\mathcal{O}_p[[X]]$. Let the Λ-adic p-ordinary cusp form F be a common eigenform of Hecke operators $T(n)$. There exists an element $\mathcal{L}_c(X,Y) \in K[[Y]]$ such that*

$$\mathcal{L}_c(\xi(u)u^{\nu+M}-1, \bar{\psi}(u)u^M-1) = \\ Qa_p(\xi, M+\nu)^{2(\alpha-\beta)}\mathcal{D}(2M+\nu-2, F(\xi, M+\nu), \bar{\psi}\bar{\xi})\Omega(\xi,k)^{-1}$$

for integers $\nu \geq 0$ and $M \geq 3$.

Proof. Our propositions 5 and 6 reduce the proof to the p-adic interpolation of the quantity

$$< F(\xi(u)u^{M+\nu}-1), \Phi(\psi\xi(u)u^{\nu}-1, \bar{\psi}(u)u^M-1) > \Omega(\xi, M+\nu)^{-1}.$$

Here $\Phi(W_1,W_2)$ is the power series in $\mathcal{O}_p[[W_1,W_2]][[q]]$ obtained as the p-adic Mellin transform of the two-variable measure μ^c constructed in proposition 5. We assume formally $1/(1+Y) = 1-Y+Y^2-Y^3+....$ Put $\Psi(X,Y) = \Phi(XY+X+Y, 1/(1+Y)) \in \mathcal{O}_p[[X,Y]][[q]]$. This change of variables transforms the interpolating quantity to the form

$$< F(\xi(u)u^{M+\nu}-1), \Psi(\xi(u)u^{M+\nu}-1, \psi(u)u^M-1) > \Omega(\xi, M+\nu)^{-1}$$

We now use the fundamental result ([4], Theorem 7.3.1, p. 209) which asserts that the space of ordinary Λ-adic forms is free of finite rank over Λ. Using [4], Theorem 7.3.6 we construct a basis in the space which consists of normalised Hecke eigenforms. All we need in order to finish the proof of the theorem is the abstract algebraic construction of the scalar product in the space of p-ordinary Λ-adic forms and its connection with the Petersson scalar product. This construction is detailly discussed in [4], p.222 (see also [5], section 1).

The possible denominators come from possible congruences among p-ordinary cusp eigenforms.

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