

A MODULAR LOOK AT FARKAS'S IDENTITIES

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ABSTRACT. Recently H. Farkas introduced a new simple arithmetic function and found an identity which involves this function. It is immediate to rewrite this identity as an identity between modular forms and reprove it in this way. We discuss natural generalizations of Farkas's identity. Surprisingly, in a certain sense, there is only one identity which is an exact analogue of that found by Farkas. At the same time, we present a way to produce infinitely many similar identities. As an application, we obtain a result on nonvanishing of the central critical value of L -function associated to a cusp Hecke eigenform.

1. STATEMENT AND DISCUSSION OF RESULTS

In a recent paper [2], Farkas introduces an arithmetic function, which he denotes by $\delta_3(n)$. This is the difference between the number of divisors of a positive integer n congruent to 1 and -1 modulo 3. As the main result of [2] (see also [3] for further discussion) he proves that for all positive integers n

$$(1) \quad \delta_3(n) + 3 \sum_{j=1}^{n-1} \delta_3(j)\delta_3(n-j) = \sigma'_3(n),$$

where

$$\sigma'_3(n) = \sum_{\substack{d|n \\ 3 \nmid d}} d.$$

We begin with a generalization of Farkas's definition. Throughout the paper, p is a prime which satisfies $p \equiv 3 \pmod{4}$, and χ is the quadratic Dirichlet character associated with $\mathbb{Q}(\sqrt{-p})$.

Define

$$(2) \quad \delta_p(n) = \sum_{d|n} \chi(d).$$

In particular, δ_3 is the function introduced by Farkas, and $\delta_7(n)$, for a positive integer n , is the difference between the number of divisors of n

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which are congruent to 1, 2 or 4 modulo 7 and the number of divisors of n which are congruent to 3, 5 or 6 modulo 7.

We also generalize Farkas's definition of $\sigma'_3(n)$ by defining for a positive integer n

$$(3) \quad \sigma'_p(n) = \sum_{\substack{d|n \\ p \nmid d}} d.$$

It is natural to rewrite Farkas's identity (1) in terms of generating functions. In order to do that we need to define $\delta_p(0)$, and this can be done in a unique way. More specifically, we prove the following.

Proposition 1. *If for a complex number $\alpha \neq 0$*

$$(4) \quad \left(\sum_{n=0}^{\infty} \delta_p(n) q^n \right)^2 = \alpha \sum_{n=0}^{\infty} \sigma'_p(n) q^n,$$

then

$$(5) \quad \delta_p(0) = -\frac{1}{2p} \sum_{a=0}^{p-1} \chi(a) a,$$

$$(6) \quad \sigma'_p(0) = \frac{p-1}{24}$$

and $\alpha = \delta_p(0)^2 / \sigma'_p(0)$.

We will use (5) along with (2) as the definition of function δ_p , and (3) along with (6) as the definition of function σ'_p .

Rewrite the formal power series identity (4) as identities for their coefficients. For $n \geq 0$ let

$$\mathcal{F}_p(n) = \sum_{j=0}^n \delta_p(j) \delta_p(n-j).$$

The identity (4) is equivalent to

$$(7) \quad \mathcal{F}_p(n) = \alpha \sigma'_p(n) \quad \text{for } n \geq 0.$$

Theorem 1. *The equivalent identities (7), (4) hold exactly in the following two cases.*

- $p = 3$. In this case $\alpha = 1/3$, and the identities reduce to the original Farkas identity (1).
- $p = 7$. In this case $\alpha = 1$.

The proof of Theorem 1 involves interpretation of $\delta_p(0)$ in terms of the class number of an imaginary quadratic field and a special case of a classical result of Minkowski. Although Theorem 1 provides an

exact analogue of Farkas's identity for $p = 7$, it is, to a large extent, a negative result. Note, however, that the identity (7) is not far from being true: the orders of magnitude of its members coincide. More specifically, we prove the following limit formula.

Theorem 2. *For $(p, n) = 1$*

$$\lim_{n \rightarrow \infty} \frac{\mathcal{F}_p(n)}{\sigma'_p(n)} = \frac{\delta_p(0)^2}{\sigma'_p(0)}.$$

Let $S_2(p)$ denote the \mathbb{C} -linear space of cusp forms of weight 2 on $\Gamma_0(p)$ with trivial Nebentypus, and let $t_p = \dim S_2(p)$. The fact that $t_p = 0$ for $p = 3$ and 7 allows to prove Theorem 1 in these cases. Although Theorem 1 claims the absence of exact analogues of Farkas's identities for the primes $p > 7$, Theorem 2 provides a hope that similar identities may still hold. This is indeed so, and we have the following generalization of Theorem 1 to the case $t_p > 0$.

Theorem 3. *There exist a set of complex numbers A_i and two sets of positive integers B_i and C_i for $i = 1, \dots, 3^{t_p}$ (all three sets depend on p) such that for any positive integer n*

$$\sum_{i=1}^{3^{t_p}} A_i \mathcal{F}_p \left(\frac{n}{B_i} C_i \right) = \sigma'_p(n),$$

where we assume

$$\mathcal{F}_p \left(\frac{n}{B} C \right) = 0$$

if n is not divisible by B .

Our proof of Theorem 3 is effective and allows to produce the claimed identities. Consider for example the first non-trivial case, $p = 11$. We have, in particular,

$$(8) \quad \mathcal{F}_{11}(19n) + 19\mathcal{F}_{11} \left(\frac{n}{19} \right) = 12\sigma'_{11}(n).$$

Theorem 3 produces in fact infinitely many choices of A_i , B_i and C_i which correspond to infinitely many identities for every $p \equiv 3 \pmod{4}$ bigger than 7 . These identities, however, have little regularity.

Finally we present a result of completely modular nature, which pertains to nonvanishing of certain central special values of L -function associated to modular forms. For a cusp Hecke eigenform $f \in S_2(p)$ with the Fourier expansion $f = \sum_{n>0} a(n)q^n$ ($q = \exp(2\pi i\tau)$ with $\Im\tau > 0$ throughout), and a Dirichlet character ψ , the associated L -series are defined as the analytic continuations of

$$L(s, f) = \sum_{n>0} a(n)n^{-s}, \quad L(s, f, \psi) = \sum_{n>0} \psi(n)a(n)n^{-s}.$$

The series satisfy a functional equation with respect to $s \mapsto 2 - s$, and the literature contains a variety of results on nonvanishing of the central special values $L(1, f, \psi)$.

Theorem 4. *For every prime $p > 7$, which satisfies $p \equiv 3 \pmod{4}$, there exists a cusp Hecke eigenform $f \in S_2(p)$ such that*

$$L(1, f)L(1, f, \chi) \neq 0$$

We remark the similarity between Theorem 4 and a corollary to a classical theorem of Kohnen and Zagier [4, Corollary 2].

2. PROOFS

For a positive integer k and a Dirichlet character ξ with $\xi(-1) = (-1)^k$, we denote by $M_k(p, \xi)$ the \mathbb{C} -linear space of (entire) modular forms of weight k on $\Gamma_0(p)$ with Nebentypus ξ , and we simply write $M_k(p)$ if k is even and the Nebentypus is trivial. We conserve the notations of the previous section, in particular, p is a prime, $p \equiv 3 \pmod{4}$, and χ is the Dirichlet character associated to $\mathbb{Q}(\sqrt{-p})$.

Proof of Proposition 1. It is well-known (see, for example, [6, XV, Th. 1.1]) that the series

$$G_{1,\chi} = -\frac{1}{2p} \sum_{a=1}^{p-1} \chi(a)a + \sum_{n \geq 1} \left(\sum_{d|n} \chi(d) \right) q^n = \sum_{n \geq 0} \delta_p(n)q^n$$

is the Fourier expansion of a modular form which belongs to $M_1(p, \chi)$. It is also well-known (see, for example, [7]) that the series

$$G_2 = \frac{p-1}{24} + \sum_{n \geq 1} \sigma'_p(n)q^n = \sum_{n \geq 0} \sigma'_p(n)q^n$$

is the Fourier expansion of a modular form which belongs to $M_2(p)$. Assume now that (4) holds with $\delta_p(0) = A$ and $\sigma'_p(0) = B$, not necessarily given by the formulas (5),(6) and some $\alpha \in \mathbb{C}$ namely,

$$(G_{1,\chi} - (\delta_p(0) - A))^2 = \alpha (G_2 - (\sigma'_p(0) - B)).$$

Rearranging terms we obtain

$$(G_{1,\chi}^2 - \alpha G_2) - 2(\delta_p(0) - A)G_{1,\chi} = -\alpha(\sigma'_p(0) - B) - (\delta_p(0) - A)^2.$$

Since the character χ is quadratic, $G_{1,\chi}^2 \in M_2(p)$, therefore, $G_{1,\chi}^2 - \alpha G_2 \in M_2(p)$. However, $2(\delta_p(0) - A)G_{1,\chi} \in M_1(p, \chi)$, and the difference of two modular forms of different weights may be a constant only if they both are zero. This establishes (5),(6). The claim about α follows from equating the constant terms in (4). □

Note that, by the Dirichlet class number formula (see, for example [1]), for $p > 3$ we have $\delta_p(0) = h_p/2$, where h_p is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$. We will use the following estimate.

Lemma 1. *If p is a prime discriminant of imaginary quadratic field (i.e. $p \equiv 3 \pmod{4}$), then*

$$h_p \leq \frac{2}{\pi} \sqrt{p} \left(\log(2\sqrt{p}/\pi) + 1 \right).$$

The authors are indebted to Boris Datskovsky for the precise formulation and proof of Lemma 1. Note that a weaker estimate $\delta_p(0) < (p-1)/12$ (and $h_p < (p-1)/6$) for $p \geq 19$, $p \equiv 7 \pmod{12}$, which is still sufficient for our applications, may be obtained directly from 5) (and the Dirichlet class number formula) by means of an elementary argument.

Proof. Let $K = \mathbb{Q}(\sqrt{-p})$, and denote by $N(X)$ the number of integral ideals \mathfrak{a} in the ring of integers of K such that their norms $\mathcal{N}(\mathfrak{a}) \leq X$.

A classical result of Minkowski [5, Th. V.4.4] specialized to the ring of integers of K states that in any ideal class there is an ideal \mathfrak{a} with its norm

$$\mathcal{N}(\mathfrak{a}) \leq 2\sqrt{p}/\pi.$$

It follows that the ideal class number

$$(9) \quad h_p \leq N(2\sqrt{p}/\pi).$$

The Dedekind ζ -function of K factors (see, for example, [5, Th. XII.1.1], [1, §V.4]) as $\zeta_K(s) = \zeta(s)L(s, \chi)$, a product of the Riemann ζ function and the Dirichlet L -function associated with χ . In other words, we have the series identity

$$\sum_{\mathfrak{a}} (\mathcal{N}(\mathfrak{a}))^{-s} = \left(\sum_{n \geq 1} n^{-s} \right) \left(\sum_{m \geq 1} \chi(m) m^{-s} \right),$$

which implies

$$(10) \quad N(X) = \sum_{mn \leq X} \chi(m) = \sum_{m \leq X} \chi(m) \left[\frac{X}{m} \right] \leq \sum_{m \leq X} \frac{X}{m} \leq X(\log X + 1),$$

where the brackets $[u]$ denote the integer part of u . The inequality asserted in Lemma 1 follows from (9) and (10). □

Proof of Theorem 1. We begin by proving (4) in the cases $p = 3$ and $p = 7$. Clearly $G_{1,\chi}^2 \in M_2(p)$. We refer to [7] for the standard facts about spaces of modular forms. In particular, recall that $M_2(p) =$

$S_2(p) \oplus E_2(p)$, where the latter summand denotes the subspace generated by Eisenstein series, which is one-dimensional since p is a prime. In both cases under consideration $\dim S_2(p) = 0$. Thus in these cases both $G_{1,x}^2$ and G_2 belong to the same one-dimensional linear space, and, therefore, must coincide up to a constant multiple, so (4) holds for $p = 3$ and $p = 7$.

Assume now that (4) holds. Equate the coefficients of q in (4) to obtain

$$2\delta_p(1)\delta_p(0) = \frac{\delta_p(0)^2}{\sigma'_p(0)}\sigma'_p(1).$$

Note that $\delta_p(1) = \sigma'_p(1) = 1$. Use (6) and $\delta_p(0) = h_p/2$ to derive the equality

$$h_p = \frac{p-1}{6}$$

The estimate of Lemma 1 implies that $h_p < (p-1)/6$ for $p > 200$, and standard class number tables show that $h_p \neq (p-1)/6$ for $7 < p \leq 199$. \square

Proof of Theorem 2. Since $G_{1,x}^2 \in M_2(p)$ and $M_2(p) = S_2(p) \oplus E_2(p)$, we may write

$$(11) \quad G_{1,x}^2 = \alpha G_2 + f$$

with $f \in S_2(p)$. Let $f = \sum_{n>0} a(n)q^n$. Equating the coefficients of q^n we obtain for $n \geq 0$

$$\mathcal{F}_p(n) = \alpha\sigma'(n) + a(n).$$

As previously, $n = 0$ provides $\alpha = \delta_p(0)^2/\sigma'_p(0)$. It follows from the definition (3) that $\sigma'(n) > n$ if $(p, n) = 1$. On the other hand the Hecke estimate for the Fourier coefficients of cusp forms [7, Cor. 2.16] implies

$$(12) \quad |a(n)| < L\sqrt{n}$$

with some constant L . \square

Proof of Theorem 3. Recall that the Hecke operator T_l for a prime $l \neq p$ acts on the space $M_2(p)$ and takes a modular form g with Fourier expansion $g = \sum b(n)q^n$ to modular form $g|T_l$ with Fourier expansion $g|T_l = \sum c(n)q^n$ with

$$c(n) = b(ln) + lb(n/l),$$

where it is assumed $c(n/l) = 0$ if $l \nmid n$. The Eisenstein series G_2 is a simultaneous eigenform of all T_l ,

$$G_2|T_l = (l+1)G_2,$$

and the linear space $S_2(p)$ admits a basis which consists of simultaneous Hecke eigenforms $g_i = \sum_{n \geq 1} c_i(n)q^n$ normalized such that $c_i(1) = 1$. Note that $g_i|T_l = c_i(l)g_i$, while the Hecke estimate (12) implies the existence of infinitely many primes l such that $c_i(l) \neq l + 1$. Now, we can rewrite (11) as an eigenform decomposition

$$(13) \quad G_{1,\chi}^2 = \alpha G_2 + \sum_{i=1}^{\dim S_2(p)} \mu_i g_i.$$

If $\mu_i = 0$ for all i , then the (13) represents the desired identity (with $t_p = 0$ and $B_1 = C_1 = 1$). However, Theorem 1 guarantees that this is not the case unless $p = 3, 7$. We are going to apply Hecke operators to both sides of (13) in order to eliminate the cusp forms g_i . Namely, pick j such that $\mu_j \neq 0$ and apply the operator $T_l - c_j(l)$ to obtain

$$G_{1,\chi}^2|T_l - c_j(l)G_{1,\chi}^2 = \alpha' G_2 + \sum_{i=1}^{\dim S_2(p)} \mu'_i g_i$$

with $\alpha' \neq 0$ and $\mu_j = 0$. Eliminating in this way, one after one, the forms g_j from the decomposition in the right, we obtain the claimed identities. □

Let us illustrate the proof of Theorem 1 with the special case $p = 11$. In this case $\dim S_2(11) = 1$, and the unique normalized cusp eigenform is $g = q \prod (1 - q^n)^2 (1 - q^{11n})^2$. We have $\delta_{11}(0) = 1/2$ and $\delta_{11}(1) = 1$, while $\sigma'_{11}(0) = 5/12$ and $\sigma'_{11}(1) = 1$. Thus (13) becomes

$$G_{1,\chi}^2 = \frac{3}{5}G_2 + \frac{2}{5}g,$$

or, equivalently,

$$\sum_{n \geq 0} \mathcal{F}(n)q^n = \frac{3}{5} \sum_{n \geq 0} \sigma'_{11}(n) + \frac{2}{5}q \prod_{n \geq 1} (1 - q^n)^2 (1 - q^{11n})^2.$$

We now apply the Hecke operator T_{19} to both sides of this identity using $g|T_{19} = 0$ and $G_2|T_{19} = 20G_2$:

$$\sum_{n \geq 0} \left(\mathcal{F}(19n) + 19\mathcal{F}(n/19) \right) = 12 \sum_{n \geq 0} \sigma'_{11}(n)q^n.$$

Equating the coefficients of q^n one obtains the sample identity (8).

Proof of Theorem 4. For $f \in S_2(p)$ with q -expansion

$$f(\tau) = \sum_{n \geq 1} b(n)q^n$$

put

$$f_\rho(\tau) = \overline{f(-\bar{\tau})} = \sum_{n \geq 1} \overline{b(n)} q^n,$$

where the bar denotes complex conjugation. It is known, that $f_\rho \in S_2(p)$, moreover, if f is a Hecke eigenform, then so is f_ρ . Clearly, $(f_\rho)_\rho = f$. Since $p > 7$, Theorem 1 implies that there exists j such that $\mu_j \neq 0$ in decomposition (13). Put $f_\rho = g_j$, and consider the Petersson scalar product of both sides of (13) with f_ρ . Since Hecke operators are Hermitian with respect to the scalar product, we have

$$\left\langle f_\rho, \alpha G_2 + \sum_{i=1}^{\dim S_2(p)} \mu_i g_i \right\rangle = \mu_j \langle f_\rho, f_\rho \rangle \neq 0$$

On the other hand, a standard application of the classical Rankin method (see, for example, [8]) implies

$$\langle f_\rho, E_{1,\chi}^2 \rangle = \Omega L(1, f) L(1, f, \chi)$$

with some $\Omega \in \mathbb{C}$. The assertion follows. (We remark that the results of Shimura [8] give an explicit expression for Ω , which we do not need here.)

□

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