Name: ________________________________

Question 1

What is the remainder when 8640^{623} is divided by 7?

<u>Answer</u>

Question 2

Prove the following well-known criterion of divisibility by 3.

A positive integer (written base 10) is divisible by 3 if an only if the sum of its digits is divisible by 3.

Question 3

For a positive integer n written in base 10, denote by $S(n)$ the sum of the digits. For example, $S(3245) = 3 + 2 + 4 + 5 = 14$, and $S(5) = 5$. Clearly, we always have the inequality $S(n) \leq n$ with an equality $S(n) = n$ if and only if $n < 10$.

Assume now that n is divisible by 9. Consider the sequence of numbers

$$S(n),\ S(S(n)),\ S(S(S(n))), \ldots,$$

where the next term is the sum of digits of the previous one. At which number does this sequence stabilize? Prove your answer.

Question 4

Let $m > 3$ be an integer. Prove that m is a prime if and only if

$$2(m-3)! \equiv -1 \pmod{m}$$

Hint Make use of Wilson's congruence and its proof.

Question 5

Let F_n denote the sequence of Fibonacci numbers. Prove the congruences for all $n > 0$:

$$F_{5n} \equiv 0 \pmod 5$$

$$F_{n+20} \equiv F_n \pmod 5$$

Question 6

A famous theorem of Dirichlet asserts that if k and l are relatively prime, then there are infinitely many primes of the form $kx + l$. The proof is rather difficult. We will address the following much weaker statement.

Let k and l be integers such that $g.c.d.(k, l) = 1$, and let $n \neq 0$ be an integer. Prove that there exists an integer x such that $g.c.d.(kx + l, n) = 1$.

Question 7

Let B_n be the $n-th$ Bernoulli number. Prove that $B_{2m+1} = 0$ for $m \geq 1$.

Question 8

Let F_n deonote the $n-th$ Fibonacci number. Since Fibonacci numbers satisfy a linear recursive relation

$$F_{n+2} = F_n + F_{n+1} \quad \text{for} \quad n \geq 1,$$

the generating function for this sequence is a rational function. Specifically, according to exercise 5 on page 44,

$$\sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2},$$

and we proved this formula in class.

Consider the sequence

$$a_n = F_n^2.$$

Hadamard's product theorem (Theorem 3 in the additional text) implies that the generating function for the sequence a_n must be rational, therefore, must satisfy a linear recurrence relation by Theorem 2 (cf. additional text). Find the linear recurrence relation for the sequence a_n.

Hint 1) Try to make use of the ideas involved into the proofs of Hadamard's product theorem.
2) the first few Fibonacci numbers are easy to calculate:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

Thus you can easily calculate first few members of our sequence a_n:

$$1, 1, 2, 9, 25, 64, 169, 441, 1156, 3025, \ldots.$$

You have to test that your linear recurrence relation indeed works. Otherwise I will test that!