Counting the relations compatible with an algebra

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Four finiteness conditions

Proof-by-picture: An easy example

Proof-by-picture: Two general conditions for ‘infiniteness’

A family of ‘finite’ examples
A compatible relation on a finite algebra $\mathbf{A}$ is a non-empty subuniverse of $\mathbf{A}^n$, for some $n \in \mathbb{N}$.

There are several natural finiteness conditions on $\mathbf{A}$ that are based on ‘how many’ compatible relations it has.
A weak finiteness condition

Condition (1): $A$ is finitely related

There is a finite set of compatible relations on $A$ from which all other compatible relations on $A$ can be defined, via primitive positive formulæ.
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- Equivalent: Clo(**A**) is determined by a finite set of relations.
- All finite lattices,¹ groups,² semilattices and unary algebras are finitely related.

¹Bergman
²Aichinger, Mayr, McKenzie
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- Equivalent: $\text{Clo}(A)$ is determined by a finite set of relations.
- All finite lattices,\(^1\) groups,\(^2\) semilattices and unary algebras are finitely related.
- Every finite commutative semigroup is finitely related,\(^3\) but not every finite semigroup.\(^4\)

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\(^2\)Aichinger, Mayr, McKenzie  
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- Every finite commutative semigroup is finitely related,\(^3\) but not every finite semigroup.\(^4\)
- The finite relatedness of \( A \) only depends on \( \text{Var}(A) \).\(^3\)

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A stronger finiteness condition

Condition (2): A has few subpowers
The logarithm of the number of $n$-ary compatible relations on A grows polynomially in $n$. 

▶ Equivalent to A having an edge term.

All finite lattices and groups have few subpowers, but not semilattices or unary algebras.

⇒: Few subpowers implies finitely related.
A stronger finiteness condition

Condition (2): $A$ has few subpowers

The logarithm of the number of $n$-ary compatible relations on $A$ grows polynomially in $n$.

- Equivalent to $A$ having an edge term.\(^5\)

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- (2) \( \Rightarrow \) (1): Few subpowers implies finitely related.\(^6\)

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An even stronger finiteness condition

Condition (3): Baker–Pixley

There is a finite set of compatible relations on $A$ from which all other compatible relations on $A$ can be defined, via conjunctions of atomic formulæ.
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There is a finite set of compatible relations on $A$ from which all other compatible relations on $A$ can be defined, via conjunctions of atomic formulæ.

▶ Equivalent to $A$ having a near-unanimity term.
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**Condition (3): Baker–Pixley**
There is a finite set of compatible relations on $A$ from which all other compatible relations on $A$ can be defined, via conjunctions of atomic formulæ.

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- All finite lattices satisfy Baker–Pixley. But not groups, semilattices or unary algebras.
- $(3) \Rightarrow (2)$: Baker–Pixley implies few subpowers.

**Condition (4): Finitely many relations**
There is a finite set of compatible relations on $A$ such that every other compatible relation on $A$ is interdefinable with one of these relations, via conjunctions of atomic formulæ.
An example

Two compatible relations on the 2-element bounded lattice $L = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle$:

$\leq \subseteq L^2$

$\rho \subseteq L^4$

The relations $\leq$ and $\rho$ are conjunct-atomic interdefinable, and so we will regard them as equivalent.

Every compatible relation on $L$ is equivalent to either $\leq$ or $\Delta_L$.
An example

Two compatible relations on the 2-element bounded lattice \( L = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle \):

\[
\leq \subseteq L^2
\]

\[
\leq = \{ (x, y) \in L^2 \mid (x, x, y, y) \in \rho \}
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An example

Two compatible relations on the 2-element bounded lattice \(L = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle\):

\[
\leq \subseteq L^2
\]

\[
\begin{array}{c}
00 \\
01 \\
11
\end{array}
\]

\[
\rho \subseteq L^4
\]

\[
\begin{array}{c}
0000 \\
0100 \\
0111 \\
1111
\end{array}
\]

\[
\leq = \{ (x, y) \in L^2 \mid (x, x, y, y) \in \rho \}
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\[
\rho = \{ (w, x, y, z) \in L^4 \mid w \leq x \& w \leq y \& y = z \}
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The relations \(\leq\) and \(\rho\) are conjunct-atomic interdefinable, and so we will regard them as equivalent.

Every compatible relation on \(L\) is equivalent to either \(\leq\) or \(\Delta_L\).
Basic definitions

Two compatible relations on $A$ are equivalent if each is conjunct-definable from the other.

If the set of all compatible relations on $A$ has only a finite number of equivalence classes, then we say that $A$ admits only finitely many relations.
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Basic definitions

Two compatible relations on \( A \) are equivalent if each is conjunct-definable from the other.

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In this case, the algebra \( A \) satisfies the Baker–Pixley condition and so \( A \) has a near-unanimity term.

Question
Which finite algebras admit only finitely many relations?
Examples

2-element Boolean algebra $\mathbb{B} = \langle \{0, 1\}; \land, \neg \rangle$

One compatible relation: $\Delta_B$. 

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2-element bounded lattice $\mathbf{L} = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle$
Two compatible relations: $\Delta_L, \leq$. 
Examples

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2-element bounded lattice $\mathbf{L} = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle$
Two compatible relations: $\Delta_L, \leq$.

2-element lattice $\mathbf{2} = \langle \{0, 1\}; \lor, \land \rangle$
Eight compatible relations: $\Delta_2, \leq, \{0\}, \{1\}, \{(0, 1)\}$,
$\leq \times \{0\}, \leq \times \{1\}, \leq \times \{(0, 1)\}$.
More examples

Finitely many relations:
- 3-element p-algebra $S = \langle \{0, d, 1\}; \lor, \land, *, 0, 1 \rangle$,
- 4-element Boolean algebra with constants,
- ring with unity $\mathbb{Z}_{pq}$, for distinct primes $p, q$,
- every quasi-primal algebra (with R. Willard),
- each finite Heyting chain (Nguyen, Pitkethly).

Infinitely many relations:
- Boolean algebras of size $\geq 4$, lattices of size $\geq 3$,
- p-algebras of size $\geq 4$, non-chain Heyting algebras, . . .
More examples

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Infinitely many relations:

- Boolean algebras of size $\geq 4$, lattices of size $\geq 3$, p-algebras of size $\geq 4$, non-chain Heyting algebras, . . .

(If $A$ admits infinitely many relations and $A \in \text{HS}(B)$, then $B$ admits infinitely many relations.)
Proof-by-picture: An easy example

Consider the 2-element algebra $\mathbf{M} = \langle \{0, 1\}; m \rangle$, where $m: \{0, 1\}^3 \rightarrow \{0, 1\}$ is the majority operation.
Proof-by-picture: An easy example

Consider the 2-element algebra $M = \langle \{0, 1\}; m \rangle$, where $m: \{0, 1\}^3 \rightarrow \{0, 1\}$ is the majority operation.

This picture proves that $M$ admits infinitely many relations:
Easy example: Steps 1 and 2

Step 1

The relation \( r := \{(0, 0), (0, 1), (1, 0)\} \) is compatible with \( M \).
Easy example: Steps 1 and 2

Step 1

The relation \( r \) defined by \((0, 0), (0, 1), (1, 0)\) is compatible with \( M \).

Define \( M = \langle \{0, 1\}; r \rangle \).
Easy example: Steps 1 and 2

Step 1
The relation \( r := \{(0, 0), (0, 1), (1, 0)\} \) is compatible with \( \mathbb{M} \).

Define \( \mathbb{M} = \langle \{0, 1\}; r \rangle \).

Step 2
Each structure \( \mathbb{X}_n \) defines a compatible relation on \( \mathbb{M} \):

\[
r_n := \text{hom}(\mathbb{X}_n, \mathbb{M}) \leq \mathbb{M}^{\mathbb{X}_n}.
\]
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For example,

\[ r_3 = \text{hom}(X_3, M) \]
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For example,

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    r_3 = \text{hom}(X_3, M) = \{(0, 0, 0), \ldots\}.
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Easy example: Steps 1 and 2

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The relation $r := \{(0, 0), (0, 1), (1, 0)\}$ is compatible with $M$.

Define $M = \langle \{0, 1\}; r \rangle$.

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Each structure $X_n$ defines a compatible relation on $M$:

$$r_n := \text{hom}(X_n, M) \subseteq M^{X_n}.$$ 

For example,

$$r_3 = \text{hom}(X_3, M) = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}.$$
Easy example: Step 3

We want to show that the relations $r_2, r_3, r_4, \ldots$ are pairwise non-equivalent.

$r_2 = \text{hom}(X_2, M)$  $r_3 = \text{hom}(X_3, M)$  $r_4 = \text{hom}(X_4, M)$
Easy example: Step 3

\[ r_2 = \text{hom}(X_2, M) \quad r_3 = \text{hom}(X_3, M) \quad r_4 = \text{hom}(X_4, M) \]

- We want to show that the relations \( r_2, r_3, r_4, \ldots \) are pairwise non-equivalent.
- We will just check that \( r_3 \) is not ca-definable from \( r_4 \).
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- We want to show that the relations \( r_2, r_3, r_4, \ldots \) are pairwise non-equivalent.
- We will just check that \( r_3 \) is not ca-definable from \( r_4 \).
- It suffices to find a map \( \varphi : Z \to \{0, 1\} \), for some \( Z \subseteq \{0, 1\}^n \), such that \( \varphi \) preserves \( r_4 \) but not \( r_3 \).
Step 3: Choosing $\varphi: Z \to \{0, 1\}$

We want $\varphi$ to preserve $r_4$ but not $r_3$. 
Step 3: Choosing $\varphi: Z \rightarrow \{0, 1\}$

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There’s always a canonical choice for $\text{dom}(\varphi)$:
Step 3: Choosing \( \varphi : \mathbb{Z} \rightarrow \{0, 1\} \)

We want \( \varphi \) to preserve \( r_4 \) but not \( r_3 \).

There’s always a canonical choice for \( \text{dom}(\varphi) \):

\[
\begin{align*}
r_3 &= \text{hom}(X_3, M) \\
&= \{(0, 0, 0), \\
&(0, 0, 1), \\
&(0, 1, 0), \\
&(1, 0, 0)\},
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\]
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$$r_3 = \text{hom}(X_3, M) = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}$$

$$Z = r_3^T = \{(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0)\}$$
Step 3: Choosing $\varphi : \mathbb{Z} \rightarrow \{0, 1\}$

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  &= \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}, \\
  Z &= r_3^T = \{(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0)\} \\
      &\quad \mapsto 1 \\
  \varphi (r_3) &= r_3 \\
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$$Z = r_3^T = \{ (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0) \} \mapsto 1,$$

So $\varphi$ doesn’t preserve $r_3$. 
Step 3: Choosing $\varphi : Z \to \{0, 1\}$

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So $\varphi$ doesn’t preserve $r_3$.

From the picture

- In fact, we have $X_3 \in \text{ISP}(M)$. Therefore $X_3 \cong Z \leq M^4$. 
Step 3: Choosing $\varphi: \mathbb{Z} \rightarrow \{0, 1\}$

We want $\varphi$ to preserve $r_4$ but not $r_3$.

There’s always a canonical choice for $\text{dom}(\varphi)$:

$$r_3 = \text{hom}(X_3, \mathcal{M})$$

$$Z = r_3^T = \{(0, 0, 0, 1), \quad \varphi \mapsto 1\}$$

$$\quad \{(0, 0, 1, 0), \quad \mapsto 1\}$$

$$\quad \{(0, 1, 0, 0)\} \rightarrow 1$$

$$\quad \{(1, 0, 0)\}\rightarrow$$

So $\varphi$ doesn’t preserve $r_3$.

From the picture

- In fact, we have $X_3 \in \text{ISP}(\mathcal{M})$. Therefore $X_3 \cong Z \leq \mathcal{M}^4$.
- The map $\varphi: X_3 \rightarrow \{0, 1\}$ does not preserve $r_3$, because it is not a graph homomorphism, i.e., $\varphi \notin \text{hom}(X_3, \mathcal{M})$. 

\[\begin{array}{c}
X_3 \\
\end{array} \xrightarrow{\varphi} \begin{array}{c}
\quad \\
\mathcal{M} \\
0 \quad 1 \quad \end{array}\]
Step 3: Checking $\varphi : \mathbb{Z} \rightarrow \{0, 1\}$ preserves $r_4$

$r_4 = \text{hom}(X_4, M) \subseteq \{0, 1\}^4$

![Diagram](image)
Step 3: Checking $\varphi : Z \to \{0, 1\}$ preserves $r_4$

$r_4 = \text{hom}(X_4, M) \subseteq \{0, 1\}^4$

Pick $w, x, y, z \in Z$. For example, $w = (0, 0, 0, 1) \xrightarrow{\varphi} 1$

$x = (0, 0, 1, 0) \xrightarrow{\varphi} 1$

$y = (0, 1, 0, 0) \xrightarrow{\varphi} 1$

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Step 3: Checking $\varphi : Z \to \{0, 1\}$ preserves $r_4$

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So $\varphi$ preserves $r_4$, since $r_4^Z = \emptyset$. 
Step 3: Checking $\varphi : Z \to \{0, 1\}$ preserves $r_4$

$r_4 = \text{hom}(X_4, M) \subseteq \{0, 1\}^4$

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So $\varphi$ preserves $r_4$, since $r_4^Z = \emptyset$.

From the picture

- Since $Z = X_3$, we are picking a map $\omega : X_4 \to X_3$. 

![Diagram](attachment:image.png)
Step 3: Checking $\varphi: \mathcal{Z} \to \{0, 1\}$ preserves $r_4$

$r_4 = \text{hom}(X_4, M) \subseteq \{0, 1\}^4$

Pick $w, x, y, z \in \mathcal{Z}$. For example, $w = (0, 0, 0, 1) \xrightarrow{\varphi} 1$

$x = (0, 0, 1, 0) \xrightarrow{} 1$

$y = (0, 1, 0, 0) \xrightarrow{} 1$

$z = (0, 0, 1, 0) \xrightarrow{} 1$

So $\varphi$ preserves $r_4$, since $r_4^Z = \emptyset$.

From the picture

- Since $\mathcal{Z} = X_3$, we are picking a map $\omega: X_4 \to X_3$.
- Want: If each $\pi_j \circ \omega : X_4 \to \{0, 1\}$ is in $r_4$, then $\varphi \circ \omega : X_4 \to \{0, 1\}$ is in $r_4$. 

Step 3: Checking $\varphi : Z \to \{0, 1\}$ preserves $r_4$

$r_4 = \text{hom}(X_4, M) \subseteq \{0, 1\}^4$

Pick $w, x, y, z \in Z$. For example, $w = (0, 0, 0, 1) \xrightarrow{\varphi} 1$

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From the picture

▶ Since $Z = X_3$, we are picking a map $\omega : X_4 \to X_3$.

▶ Want: If each $\pi_i \circ \omega : X_4 \to \{0, 1\}$ is in $r_4$, then $\varphi \circ \omega : X_4 \to \{0, 1\}$ is in $r_4$.

▶ But $X_3 \leq M^4$, so we really want: If $\omega : X_4 \to X_3$, then $\varphi \circ \omega : X_4 \to M$ is a graph homomorphism. True vacuously.
Proof-by-picture: Recap

Consider the 2-element algebra \( M = \langle \{0, 1\}; m \rangle \), where \( m: \{0, 1\}^3 \to \{0, 1\} \) is the majority operation.

This picture below proves that \( M \) admits infinitely many relations.

The red labels on \( X_n \) yield a map \( \varphi_n: X_n \to \{0, 1\} \) that is not a graph homomorphism.
Proof-by-picture: General approach

**Lemma**

To show that \( A \) admits infinitely many relations, it suffices to find

- a structure \( \mathbb{A} = \langle A; R \rangle \) that is compatible with \( A \), and skip
Proof-by-picture: General approach

Lemma

To show that $A$ admits infinitely many relations, it suffices to find

- a structure $\mathbb{A} = \langle A; R \rangle$ that is compatible with $A$, and skip
- for all $n \in \mathbb{N}$,
  - a finite structure $X_n \in \text{ISP}(A)$, and
  - a map $\varphi_n : X_n \to A$ that is not a morphism from $X_n$ to $A$
Lemma

To show that \( A \) admits infinitely many relations, it suffices to find

- a structure \( A = \langle A; R \rangle \) that is compatible with \( A \), and
- for all \( n \in \mathbb{N} \),
  - a finite structure \( X_n \in \text{ISP}(A) \), and
  - a map \( \varphi_n : X_n \rightarrow A \) that is not a morphism from \( X_n \) to \( A \)

such that, either for all \( k < \ell \) or for all \( k > \ell \), the following condition holds:

- for every morphism \( \omega : X_\ell \rightarrow X_k \), the map \( \varphi_k \circ \omega : X_\ell \rightarrow A \) is a morphism from \( X_\ell \) to \( A \) (i.e., the relation \( r_k \) is not ca-definable from \( r_\ell \)).
A general condition for ‘infiniteness’

A bad relation

If a finite algebra $A$ either

- has a pair of non-permuting congruences, or
- is of the form $B^2$, for non-trivial $B$, 

For non-permuting congruences:

$$r = \theta_1 \cdot \theta_2.$$ 

For $A = B^2$:

$$r = \{ (x_1, y_1), (x_2, y_2) \in A^2 | x_2 = y_1 \}.$$ 

A general condition for ‘infiniteness’

A bad relation

If a finite algebra $\mathbf{A}$ either
  - has a pair of non-permuting congruences, or
  - is of the form $\mathbf{B}^2$, for non-trivial $\mathbf{B}$,
then $\mathbf{A}$ has elements $a, b, c$ and a compatible binary relation $r$ such that

For non-permuting congruences:
$$r = \theta_1 \cdot \theta_2.$$ 

For $\mathbf{A} = \mathbf{B}^2$:
$$r = \{ (x_1, y_1), (x_2, y_2) \in \mathbf{A}^2 \mid x_2 = y_1 \}.$$ 

\[ \text{in } r \quad \begin{array}{ccc} \circ & a & \circ \\ \circ & b & \circ \end{array} \quad \text{not in } r \quad \begin{array}{ccc} \circ & a & \circ \\ \circ & b & \circ \end{array} \]
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For non-permuting congruences: $r = \theta_1 \cdot \theta_2$.

For $A = B^2$: $r = \{ ((x_1, x_2), (y_1, y_2)) \in A^2 \mid x_2 = y_1 \}$. 
A general condition for ‘infiniteness’

Theorem

Let $A$ be a finite algebra with elements $a, b, c$ and a compatible binary relation $r$ such that

Then $A$ admits infinitely many relations.
A general condition for ‘infiniteness’

The theorem

**Theorem**

Let $A$ be a finite algebra with elements $a, b, c$ and a compatible binary relation $r$ such that

$$\text{in } r \quad c \quad a \quad b \quad \text{not in } r \quad c \quad a$$

Then $A$ admits infinitely many relations.

There is an obvious choice for $A = \langle A; R \rangle$, 
A general condition for ‘infiniteness’

Theorem

Let $A$ be a finite algebra with elements $a, b, c$ and a compatible binary relation $r$ such that

- $c \circ a \in r$
- $c \circ a \not\in r$
- $b \circ a$

Then $A$ admits infinitely many relations.

There is an obvious choice for $A = \langle A; R \rangle$, namely, $A := \langle A; r \rangle$. 
Theorem

Let $A$ be a finite algebra with elements $a, b, c$ and a compatible binary relation $r$ such that

Then $A$ admits infinitely many relations.

Proof of Case 1: $(b, b) \notin r$ or $(c, c) \notin r$
A general condition for ‘infiniteness’

The theorem

**Theorem**

Let \( A \) be a finite algebra with elements \( a, b, c \) and a compatible binary relation \( r \) such that

\[
in r \quad \begin{cases} a & \text{c} \\ b & \text{c} \end{cases} \quad \text{not in } r \quad \begin{cases} c & \text{a} \\ c & \text{b} \end{cases}
\]

Then \( A \) admits infinitely many relations.

Proof of Case 2: \((b, b) \in r \) and \((c, c) \in r\)
A general condition for ‘infiniteness’

Applications

Corollary

If one of the following holds, then \( A \) admits infinitely many relations:

1. \( A \) has a pair of non-permuting congruences;
2. \( A \) is isomorphic to \( B^2 \), with \( B \) non-trivial;
3. \( A \) is a subalgebra of \( B^2 \) such that \( \{(0, 0), (0, 1), (1, 0)\} \subseteq A \), for some \( 0 \neq 1 \) in \( B \).

This result covers:

- Boolean algebras of size \( \geq 4 \),
- lattices of size \( \geq 3 \),
- \( p \)-algebras of size \( \geq 4 \),
- non-chain Heyting algebras, \ldots
Another general condition for ‘infiniteness’

Theorem and example

**Theorem**

*If $\mathbf{A}$ admits only finitely many relations, then $\text{Con}(\mathbf{A})$ is an $\mathcal{N}$-free distributive lattice.*
Another general condition for ‘infiniteness’

Theorem and example

**Theorem**

*If* $A$ *admits only finitely many relations, then* $\text{Con}(A)$ *is an* $\mathbb{N}$*-free distributive lattice.*

**Example**

Assume $M = A \times B \times C$, for non-trivial finite algebras $A, B, C$. Then $M$ admits infinitely many relations.
Another general condition for ‘infiniteness’

Proof-by-picture

Proof

Assume \( \text{Con}(A) \) contains \( \alpha \).
Another general condition for ‘infiniteness’

Proof-by-picture

Proof

Assume \text{Con}(A) contains \begin{tikzpicture}[baseline=(current bounding box.center)]
\node (a) at (0,0) {$\alpha$};
\node (b) at (1,0) {$\beta$};
\node (c) at (1,1) {$\delta$};
\draw (a) -- (b);
\draw (b) -- (c);
\end{tikzpicture}.

Take \( A = \langle A; \alpha, \beta, \delta \rangle \) and choose \((a, b) \in \delta \setminus \beta\).
Another general condition for ‘infiniteness’

Proof-by-picture

Proof

Assume $\text{Con}(A)$ contains $\alpha, \beta, \delta$.

Take $A = \langle A; \alpha, \beta, \delta \rangle$ and choose $(a, b) \in \delta \setminus \beta$.

$X_4$ $\rightarrow$ $X_6$ $\rightarrow$ $X_8$
Another general condition for ‘infiniteness’

Proof-by-picture

Proof

Assume $\text{Con}(A)$ contains $\alpha \land \beta$.

Take $A = \langle A; \alpha, \beta, \delta \rangle$ and choose $(a, b) \in \delta \setminus \beta$.

So $A$ admits infinitely many relations.
Finding examples of ‘finiteness’

A definition

A finite algebra $A$ is **strictly affine complete** if:

for all $n \in \mathbb{N}$ and all $X \subseteq A^n$, every function $f : X \to A$ that preserves each $\theta \in \text{Con}(A)$ extends to a polynomial of $A$. 

Example

Every finite algebra that generates an arithmetical variety is strictly affine complete (Pixley).
A finite algebra $A$ is strictly affine complete if:

for all $n \in \mathbb{N}$ and all $X \subseteq A^n$, every function $f : X \to A$ that preserves each $\theta \in \text{Con}(A)$ extends to a polynomial of $A$.

Example
Every finite algebra that generates an arithmetical variety is strictly affine complete (Pixley).
Theorem

Let $A$ be a finite algebra such that

1. $A$ is strictly affine complete,
2. each $a \in A$ is the value of a constant term function of $A$, and
3. $\text{Con}(A)$ is an $\mathbb{N}$-free lattice.

Then $A$ admits only finitely many relations.
Finding examples of ‘finiteness’
A theorem

**Theorem**

*Let $A$ be a finite algebra such that*

1. $A$ is strictly affine complete,
2. each $a \in A$ is the value of a constant term function of $A$, and
3. $\text{Con}(A)$ is an $\mathbb{N}$-free lattice.

*Then $A$ admits only finitely many relations.*

**Proof**

We use the duality given by the alter ego $\mathbb{A} := \langle A; \text{Con}(A), \mathcal{F} \rangle$ to represent all the compatible relations on $A$. 
Finding examples of ‘finiteness’
Application of the theorem

Example
Let $L$ be a finite $\mathbb{N}$-free distributive lattice.
There exists a finite algebra $A$ with $\text{Con}(A) \cong L$ such that $A$ admits only finitely many relations.
Finding examples of ‘finiteness’
Application of the theorem

Example
Let $L$ be a finite $\mathbb{N}$-free distributive lattice.
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Proof
Finding examples of ‘finiteness’

Application of the theorem

Example
Let $L$ be a finite $\mathbb{N}$-free distributive lattice.
There exists a finite algebra $A$ with $\text{Con}(A) \cong L$ such that $A$ admits only finitely many relations.

Proof

$L = \langle A; \lor, \land, \rightarrow, a_1, a_2, \ldots, a_n \rangle$
Example

Assume that $\mathbf{M} = \mathbf{A} \times \mathbf{B}$ is the independent product of two primal algebras. Then $\mathbf{M}$ admits only finitely many relations.
Finding examples of ‘finiteness’
Another application of the theorem

Example

Assume that $M = A \times B$ is the independent product of two primal algebras. Then $M$ admits only finitely many relations.

For example, the following algebras admit only finitely many relations:

- the four-element Boolean algebra enriched with all elements as constants;
- the ring with unity $\mathbb{Z}_{pq}$, for all distinct primes $p$ and $q$. 
Finding examples of ‘finiteness’
Another application of the theorem

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Mahalo!