Separating Clones Near the Top of the Clone Lattice

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AMS Sectional Meeting
Honolulu, HI, March 3–4, 2012
A finite, \( k := |A| \geq 2 \)
Separation Theorem for a Clone $Q$

$A$ finite, $k := |A| \geq 2$

$\mathcal{P} := \mathcal{L}_A - [Q)$

$\mathcal{P} \supseteq \mathcal{O}_A$

$\mathcal{P}_{\text{max}}$
A finite, \( k := |A| \geq 2 \)

\[ \mathcal{P} := \mathcal{L}_A - [Q] \]

\( \mathcal{Q} \text{ fin gen} \Rightarrow \bullet \ Q \nsubseteq C \iff C \subseteq \mathcal{M} \text{ for some } \mathcal{M} \in \mathcal{P}_{\text{max}} \)
Separation Theorem for a Clone $\mathcal{Q}$

A finite, $k := |A| \geq 2$

$\mathcal{P} := \mathfrak{S}_A - [\mathcal{Q}]$

$\mathcal{Q}$ fin gen $\iff$ $\bullet \mathcal{Q} \not\subseteq \mathcal{C} \iff \mathcal{C} \subseteq \mathcal{M}$ for some $\mathcal{M} \in \mathcal{P}_{\text{max}}$

$[\mathcal{Q}]$ finite $\implies$ $\bullet \mathcal{P}_{\text{max}}$ is finite
Example 1: The Maximal Subclones of $\mathcal{O}_A$

\[ \mathcal{Q} = \mathcal{O}_A \]

**Rosenberg’s Thm.** The maximal clones on $A$ are of the clones \( \{\rho\}^\perp = \{f \in \mathcal{O}_A : f \text{ preserves } \rho\} \) for one of six types of rels $\rho$: 

- Affn
- Centr
- Eq
- Perm
- Reg
- BPO
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Example 2: Submaximal Clones

$Q = \{\rho\}^\perp$, a finitely generated maximal clone on $A$
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Example 2: Submaximal Clones

\[ Q = \{ \rho \}^\perp, \text{ a finitely generated maximal clone on } A \]

Submax clones known if \( \rho \) is

- permutation [Rosenberg–Sz]
- subset [Lau]
- affine relation [Sz]
- equivalence relation with one nontrivial block [Lau]
- \( \iota \), i.e., \( Q \) is Słupecki's clone [Rosenberg–Haddad]

If \( |A| = k = 3 \) [Lau].

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\[ \mathcal{L}_A \]
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or if \( |A| = k = 3 \) [Lau].
Name your favorite \( \mathcal{Q} \)

\[\mathcal{O}_A \]

\[\mathcal{P}_{\text{max}}\]

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Separating Clones Near the Top of the Clone Lattice
Name your favorite $Q$

$\diamond\ Q = \text{clone generated by the ternary discriminator}$
\( \mathcal{Q} = \) clone generated by the ternary discriminator
\( \mathcal{Q} = \) clone of nonsurjective operations:
\[ S^- := \{ f \in \mathcal{O}_A : f \text{ nonsurj} \} \cup \langle \text{id} \rangle, \]
$Q = \text{clone generated by the ternary discriminator}$

$Q = \text{clone of nonsurjective operations:}$

$S^- := \{ f \in \mathcal{O}_A : f \text{ nonsurj} \} \cup \langle \text{id} \rangle,$

a subclone of Słupecki’s clone:

$S := \{ f \in \mathcal{O}_A : f \text{ nonsurj} \} \cup \langle \text{Sym}_A \rangle$
\[ Q = \text{clone generated by the ternary discriminator} \]
\[ Q = \text{clone of nonsurjective operations:} \]
\[ S^- := \{ f \in \mathcal{O}_A : f \text{ nonsurj} \} \cup \langle \text{id} \rangle, \]
\[ \text{a subclone of Śłupecki’s clone:} \]
\[ S := \{ f \in \mathcal{O}_A : f \text{ nonsurj} \} \cup \langle \text{Sym}_A \rangle \]
\[ \ldots \]
$\equiv_C$ on $\mathcal{O}_A$

$f \equiv_C g \iff f = g(h_1, \ldots)$ and $g = f(h'_1, \ldots)$ for some $h_1, \ldots, h'_1, \ldots \in C$. 
$\equiv_C$ on $\mathcal{O}_A$:

$f \equiv_C g \iff f = g(h_1, \ldots)$ and $g = f(h'_1, \ldots)$ for some $h_1, \ldots, h'_1, \ldots \in \mathcal{C}$.

[Lehtonen–Sz]
\( \equiv_C \) on \( \mathcal{O}_A \):
\[ f \equiv_C g \iff f = g(h_1, \ldots) \text{ and } g = f(h_1', \ldots) \text{ for some } h_1, \ldots, h_1', \ldots \in \mathcal{C}. \]

[Lehtonen–Sz]

\( (k-1) \)-ary central rels
\( \sigma_1, \sigma_k \)
\{1\}, \{k\}
sub-sets
perms
equiv. rels
\( \mathcal{O}_A \)

\( \mathcal{C}_E \) determined by:
chain \( E \) of equiv rels & perms preserving \( E \) & all subsets
least discriminator clone (\( E \) triv)

Filter of clones \( \mathcal{C} \) with fin many \( \equiv_C \)-classes on \( \mathcal{O}_A \)
\( \equiv_C \) on \( \mathcal{O}_A \):

\[ f \equiv_C g \iff f = g(h_1, \ldots) \text{ and } g = f(h'_1, \ldots) \text{ for some } h_1, \ldots, h'_1, \ldots \in \mathcal{C}. \]

[Lehtonen–Sz]
$\equiv_C$ on $\mathcal{O}_A$:

$f \equiv_C g \iff f = g(h_1, \ldots)$ and $g = f(h'_1, \ldots)$ for some $h_1, \ldots, h'_1, \ldots \in \mathcal{C}$.

$\sigma_1 \ldots \sigma_k$

least discriminator clone ($E$ triv)

$C_E$ determined by:

chain $E$ of equiv rels
& perms preserving $E$
& all subsets

Filter of clones $\mathcal{C}$
with fin many
$\equiv_C$-classes on $\mathcal{O}_A$

Sep Thm for $S^-$
$
\downarrow
$
no other clones in Filter here
$\equiv_C$ on $\mathcal{O}_A$: $\mathcal{F}(A, U) := \{A^n \overset{f}{\to} U : n \geq 1\}$:

$f \equiv_C g \iff f = g(h_1, \ldots)$ and $g = f(h'_1, \ldots)$ for some $h_1, \ldots, h'_1, \ldots \in \mathcal{C}$. 
$\equiv_C$ on $\mathcal{O}_A$: $\mathcal{F}(A, U) := \{ A^n \rightarrow^f U : n \geq 1 \}$:

$f \equiv_C g \iff f = g(h_1, \ldots)$ and $g = f(h'_1, \ldots)$ for some $h_1, \ldots, h'_1, \ldots \in \mathcal{C}$.

If $|U| > |A| + 1$:
Let $|A| = k > 2$. 

$\mathcal{V} := \mathcal{L}_A - [S^-)$
Let $|A| = k > 2$.

$$\mathcal{P} := \mathcal{L}_A - [S^-]$$
Subclones of $S$: Some Facts

$2^{\aleph_0}$ subclones,

$S = S^{<k}$
Subclones of $S$: Some Facts

$|2^\aleph_0|$ subclones,

\[
S = S^{<k} = S^{<k-1} = S^{<3} = \langle T \rangle
\]

(Słupecki, 1939; Burle, 1967)
Subclones of $S$: Some Facts

$2^\aleph_0$ subclones,

\[
S = S^{<k}
\]

(Słupecki, 1939; Burle, 1967)

(Szabó, unpubl.)

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2^\aleph_0 subclones,

\( S = S^<_k \)

\( S^<_k - (S^<_{k-1})^- \)

\( S^<_3 \)

\( S^<_3 - (S^<_3)^- \)

\( S^<_3 - (S^<_3)^- \)

\( \langle T^- \rangle \)

\( \langle Sym_A \rangle \)

(Haddad–Rosenberg, 1994)

(Słupecki, 1939; Burle, 1967)

(Szabó, unpubl.)

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Separating Clones Near the Top of the Clone Lattice
Subclones of $S$: Some Facts

$2^\mathbb{N}_0$ subclones, finitely many maximal subclones

(Szláveczki, 1939; Burle, 1967)

(Szabó, unpubl.)

(Haddad–Rosenberg, 1994)
Corollary

[Haddad–Rosenberg]

*The maximal subclones of $S$ are the following:*

$$S_{k-1} := \{ f \in S : f(x, \ldots, x) \in \text{Sym} A \Rightarrow f(x, \ldots, x) \in G \}$$

where $G$ is a maximal subgroup of $\text{Sym} A$, $S_{k-1} < S$ if $k \geq 4$, and $B$ if $k = 3$. 
The Maximal Subclones of $S$

Corollary

[Haddad–Rosenberg]

The maximal subclones of $S$ are the following:

$$S[G] := \{ f \in S : f(x, \ldots, x) \in \text{Sym}_A \Rightarrow f(x, \ldots, x) \in G \}$$

where $G$ is a maximal subgroup of $\text{Sym}_A$, and $S < k - 1$ if $k \geq 4$, and $B$ if $k = 3$. 
Corollary

[Haddad–Rosenberg]
The maximal subclones of $S$ are the following:

- $S[G] := \{ f \in S : f(x, \ldots, x) \in \text{Sym}_A \Rightarrow f(x, \ldots, x) \in G \}$
  where $G$ is a maximal subgroup of $\text{Sym}_A$,

- $S^{<k-1}$ if $k \geq 4$, and $B$ if $k = 3$. 

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Separating Clones Near the Top of the Clone Lattice
Theorem

The maximal members of $\mathcal{L}_A - [S^-]$ are the clones $\{\rho\} \perp$ with

Here:
- $\text{Reg}^*$ := $\text{Reg} - \{\iota_k\}$
- $\{\iota_k\} \perp S$
The maximal members of $\mathcal{L}_A - [S^-)$ are the clones $\{\rho\} \perp$ with $\rho \in \text{BPO} \cup \text{Perm} \cup \text{Affn} \cup \text{Eq} \cup \text{Centr} \cup \text{Reg}^*$

Here:
- $\text{Reg}^* := \text{Reg} - \{\iota_k\}$
- $\{\iota_k\} \perp = S$
- $\{\beta\} \perp = \text{Burle's clone}$
The maximal members of \( \mathcal{L}_A - [S^-] \) are the clones \( \{\rho\} \perp \) with

\[
\rho \in \text{BPO} \cup \text{Perm} \cup \text{Affn} \cup \text{Eq} \cup \text{Centr} \cup \text{Reg}^* \cup \begin{cases} 
\text{aCentr} \cup \text{aReg} & \text{if } k \geq 4, \\
\{\beta\} & \text{if } k = 3.
\end{cases}
\]
Theorem

The maximal members of \( \mathcal{L}_A - [S^-] \) are the clones \( \{\rho\} \perp \) with

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\rho \in \text{BPO} \cup \text{Perm} \cup \text{Affn} \cup \text{Eq} \cup \text{Centr} \cup \text{Reg}^* \\
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Here:

- \( \text{Reg}^* := \text{Reg} - \{\iota_k\}, \quad \{\iota_k\} \perp = S \)
The maximal members of $\mathcal{L}_A - [S^-]$ are the clones $\{\rho\}^\perp$ with

$$\rho \in BPO \cup Perm \cup Affn \cup Eq \cup Centr \cup Reg^*$$

$$\cup \begin{cases} aCentr \cup aReg & \text{if } k \geq 4, \\ \{\beta\} & \text{if } k = 3. \end{cases}$$

Here:

- $Reg^* := Reg - \{\iota_k\}$, $\{\iota_k\}^\perp = S$
- $\{\beta\}^\perp = B = \text{Burle's clone}$
For $1 \leq m \leq k = |A|$, let

$$\iota_m := \{ \bar{a} \in A^m : a_i = a_j \text{ for some } i \neq j \}.$$
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**Definition**

An $m$-ary relation $\rho$ on $A$ is

- **central** if $1 \leq m \leq k - 1$, 
- **almost central** if $2 \leq m \leq k - 2$ and $\rho \neq A^m$, but for each $B \in (A^{k-1})$, either $\rho|_B$ is central on $B$ or $\rho|_B = B^m$. 

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**Definition**

An $m$-ary relation $\rho$ on $A$ is

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  - $\rho$ is totally symmetric: invariant under permuting coords,
  - and

- **almost central** if $2 \leq m \leq k - 2$ and $\rho \neq A^m$ and for each $B \in (A^{k-1})$ either $\rho|_B$ is central on $B$ or $\rho|_B = B^m$.
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- **central** if $1 \leq m \leq k - 1$,
  - $\rho$ is totally reflexive: $\iota_m \subseteq \rho$,
  - $\rho$ is totally symmetric: invariant under permuting coords, and
  - $\rho \neq A^m$, but $\{c\} \times A^{m-1} \subseteq \rho$ for some $c \in A$. 

- **almost central** if $2 \leq m \leq k - 2$ $\rho \neq A^m$ and $\rho$ is not a central rel, but for each $B \in (A^{k-1})$, either $\rho|_B$ is central on $B$ or $\rho|_B = B^m$. 

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For $1 \leq m \leq k = |A|$, let

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- **almost central** if $2 \leq m \leq k - 2$
  - $\rho \neq A^m$ and $\rho$ is not a central rel, but
  - for each $B \in \binom{A}{k-1}$, either $\rho|_B$ is central on $B$ or $\rho|_B = B^m$. 

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Separating Clones Near the Top of the Clone Lattice
Definitions: Regular and Almost Regular Relations

For a set $E$ of equiv rels on $A$, with exactly $m$ blocks each, let

$$\lambda_E := \bigcap_{\theta \in E} \lambda_\theta \quad \text{where} \quad \lambda_\theta := \{ \overline{a} \in A^m : a_i \theta a_j \text{ for some } i \neq j \}.$$

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For a set $E$ of equiv rels on $A$, with exactly $m$ blocks each, let

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$$

**Definition**

An $m$-ary relation $\rho$ on $A$ is

- **Regular** if $3 \leq m \leq k$ and $\rho = \lambda_E$ with $E$ as above s.t. $\bigcap_{r \in E} \bigcap_{\ell \in E} \neq \emptyset$ for any blocks $B_\ell$ of $\theta_\ell$.

- **Almost Regular** if either $3 \leq m \leq k - 2$ and $\rho = \lambda_E$ with $E$ as above, $|E| \geq 2$, s.t. $B_j \cap B_\ell = \emptyset$ for nonsingl. blocks $B_j$ of $\theta_j$, $B_\ell$ of $\theta_\ell$ with $j \neq \ell$; or

  - $m = k - 1 \geq 3$ and $\rho$ is totally reflexive, totally symmetric, $\rho \notin \text{Centr} \cup \text{Reg}^*$.
Definitions: Regular and Almost Regular Relations

For a set $E$ of equiv rels on $A$, with exactly $m$ blocks each, let

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\lambda_E := \bigcap_{\theta \in E} \lambda_\theta \quad \text{where} \quad \lambda_\theta := \{ \overline{a} \in A^m : a_i \theta a_j \text{ for some } i \neq j \}.
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An $m$-ary relation $\rho$ on $A$ is

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**Definition**

An $m$-ary relation $\rho$ on $A$ is

- **regular** if $3 \leq m \leq k$ and $\rho = \lambda_E$ with $E$ as above s.t.
  - $\bigcap_{\ell=1}^r B_\ell \neq \emptyset$ for any blocks $B_\ell$ of $\theta_\ell$.

- almost regular if either $3 \leq m \leq k - 2$ and $\rho = \lambda_E$ with $E$ as above, $|E| \geq 2$, s.t.
  - $B_j \cap B_\ell = \emptyset$ for nonsingl. blocks $B_j$ of $\theta_j$, $B_\ell$ of $\theta_\ell$ with $j \neq \ell$;
  - or $m = k - 1 \geq 3$ and $\rho$ is totally reflexive, totally symmetric, $\rho \not\in \text{Centr} \cup \text{Reg}^*$. 

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For a set $E$ of equiv rels on $A$, with exactly $m$ blocks each, let

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Definition

An $m$-ary relation $\rho$ on $A$ is

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- **almost regular** if
  - either $3 \leq m \leq k - 2$ and $\rho = \lambda_E$ with $E$ as above, $|E| \geq 2$, s.t.
Definitions: Regular and Almost Regular Relations

For a set $E$ of equiv rels on $A$, with exactly $m$ blocks each, let

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**Definition**

An $m$-ary relation $\rho$ on $A$ is

- **regular** if $3 \leq m \leq k$ and $\rho = \lambda_E$ with $E$ as above s.t. $\bigcap_{\ell=1}^{r} B_\ell \neq \emptyset$ for any blocks $B_\ell$ of $\theta_\ell$.

- **almost regular** if

  either $3 \leq m \leq k - 2$ and $\rho = \lambda_E$ with $E$ as above, $|E| \geq 2$, s.t.

  $B_j \cap B_\ell = \emptyset$ for nonsingl. blocks $B_j$ of $\theta_j$, $B_\ell$ of $\theta_\ell$ with $j \neq \ell$;

  or

  $m = k - 1 \geq 3$ and $\rho$ is totally reflexive, totally symmetric, $\rho \not\in \text{Centr} \cup \text{Reg}^*$. 

Á. Szendrei
Definitions: Regular and Almost Regular Relations

For a set $E$ of equiv rels on $A$, with exactly $m$ blocks each, let

$$\lambda_E := \bigcap_{\theta \in E} \lambda_\theta$$

where

$$\lambda_\theta := \{ \bar{a} \in A^m : a_i \theta a_j \text{ for some } i \neq j \}.$$

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The Maximal Subclones of $\mathcal{U}$, $S^- \subseteq \mathcal{U} \subseteq S$

$S^- \subseteq \mathcal{U} \subseteq S \implies \mathcal{U} = S[G]$ for some subgroup $G \subseteq \text{Sym}_A$

Corollary

Every maximal subclone of $S[G]$ is of the form

1. $S[H]$ for a maximal subgroup $H$ of $G$, or
2. $S \cap \{\rho\} \perp$ for some

$$\rho \in \text{BPO} \cup \text{Eq} \cup \text{Centr} \cup \text{Reg}^* \cup \begin{cases} \text{aCentr} \cup \text{aReg} & \text{if } k \geq 4, \\ \{\beta\} & \text{if } k = 3 \end{cases}$$

such that $G \subseteq \{\rho\} \perp$.

For some $\rho \in \text{BPO}, \text{Centr}, \text{Reg}^*$ in (2), $S \cap \{\rho\} \perp$ is not max in $S[G]$. 