Approximate or non-continuous satisfaction of identities

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Six sections of the talk.

**Basics**

Some calculations on approximate satisfaction. $\lambda(A, \Sigma)$.

Precise definitions for discontinuous satisfaction. $\mu_n(A, \Sigma)$.

An example: $\mu_1(A, \Sigma) = 0; \mu_2(A, \Sigma) = \text{diam}(A)$.

Some further results

Algorithmic questions about $\lambda$ and $\mu_n$. 
Compatibility: $A \models \Sigma$
Given a topological space $A$ and a set $\Sigma$ of equations in operation symbols $F_t$, we write

$$A \models \Sigma,$$

and say that $A$ and $\Sigma$ are compatible, iff there exist continuous operations $\overline{F}_t$ on $A$ satisfying $\Sigma$. 

Examples: Groups on $S^1$, $S^3$ and $\mathbb{R}$, various matrix groups, many H-spaces, a lattice on $[0,1]$, a ternary median operation on $Y$, simple $\Sigma$ on absolute-retract $A$, Sets $\left[ \begin{array}{c} n \end{array} \right]$ on any space $A^n$, a unital ring on $S^1 \times \mathbb{Z}$, a Boolean algebra on $\{0,1\}^{\aleph_0}$. Even, for any $A$ and $\Sigma$, the Swierczkowski free algebra $F_A(\Sigma)$ (whose universe is a superset of $A$).
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A \models \Sigma is a mysterious, sparse relation

Experientially, A \models \Sigma occurs only sporadically, whereas it can in many cases be proved false. But A \not\models \Sigma does not seem to have any uniform method of proof. E.g. the sphere $S^n \models \Sigma$ only for trivial $\Sigma$ or for $n = 1, 3, 7$. (Hard algebraic topology to prove this.) There is no algorithm that settles $R \models \Sigma$ for finite $\Sigma$. (Uses Matiasevich solution of Hilbert's Tenth Problem.) Thus $\models$ is not too sparse.
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Approximate replacements for $A \models \Sigma$
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For $(A, d)$ a metric space, and $\eta > 0$

$$A \models_{\eta} \Sigma,$$

will mean that there exist *continuous* operations $F_t$ on $A$ satisfying $\Sigma$ *within* $\eta$. (Of course we may also study $A \models_{\epsilon} \eta \Sigma$!)
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Some further results

Algorithmic questions about $\lambda$ and $\mu_n$. 
Some review of $A \models_{\eta} \Sigma$

Our long paper, "Approximate satisfaction of identities," deals with $\models_{\eta}$, and we have spoken on it before, but we will sketch one result for later comparison with $\models_{\varepsilon}$.

For arbitrary metric space $A$ and equations $\Sigma$, we define the real number (or $+\infty$):

$$\lambda(A, \Sigma) = \inf\{\eta : A \models \eta \Sigma\}.$$
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Σ saying, “$G$ is a one-one binary operation on $A$.”
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Σ will be this pair of equations:

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F_0(G(x_0, x_1)) \approx x_0, \quad F_1(G(x_0, x_1)) \approx x_1.
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A^2 \xrightarrow{\overline{G}} A \xrightarrow{\overline{F}} A^2 = \text{identity},
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where \( \overline{F} \) has \( \overline{F}_0 \) and \( \overline{F}_1 \) as its component functions.
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where $\overline{F}$ has $\overline{F}_0$ and $\overline{F}_1$ as its component functions. Thus under exact satisfaction, $\overline{G}$ must be one-one and $\overline{F}$ must be onto.
\( \lambda([0, 1], \Sigma) = 0.5 \)
First, an easy exercise yields $\lambda([0, 1], \Sigma) \leq 0.5$.

For the opposite inequality, we consider a topolgical algebra $(A; \overline{G}, \overline{F_0}, \overline{F_1})$ modeling $\Sigma$ within $K$; we will show that $K \geq 0.5$. 

Let $\{a_0, a_1\} = \{b_0, b_1\} = \{0, 1\}$. Compare the four values $G(a_i, b_j)$; w.l.o.g. $G(a_0, b_0)$ is the smallest. Again w.l.o.g. we have $G(a_1, b_0) \leq G(a_0, b_1)$. In other words, $G(a_0, b_0) \leq G(a_1, b_0) \leq G(a_0, b_1)$. We consider the real function $H(x) = G(a_0, x)$. By the IVT there exists $e$ with $G(a_0, e) = G(a_1, b_0)$. 

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\( \lambda([0, 1], \Sigma) = 0.5 \), continued

To repeat, we now have \( e \) with

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From the approximate validity of \( \Sigma \), we now calculate

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Since \( \{a_0, a_1\} = \{0, 1\} \), we now have the desired conclusion that

\[
1 = d(a_0, a_1) \leq 2K.
\]
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Given real \( \eta > 0 \), we construct a unit-diameter metric \( d \) for \([0, 1]^2\) such that \( \lambda(([0, 1]^2, d), \Sigma) \leq \eta. \)
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We replace \([0, 1]^2\) by \( B = [0, \sqrt{1 - \eta^2}] \times [0, \eta] \), while taking \( d \) to be the Euclidean metric. Clearly \( B \cong [0, 1]^2 \), and moreover \( B \) has unit diameter.
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\[
\begin{align*}
\overline{G}((a_0, a_1), (b_0, b_1)) &= (a_0, b_0/K) \\
\overline{F}_0(a_0, a_1) &= (a_0, 0) \\
\overline{F}_1(a_0, a_1) &= (Ka_1, 0).
\end{align*}
\]

where \(K = \sqrt{1-\eta^2}/\eta.\)
We now calculate

\[
\bar{F}_0(\bar{G}((a_0, a_1), (b_0, b_1))) = (a_0, 0)
\]

\[
d((a_0, a_1), \bar{F}_0(\bar{G}((a_0, a_1), (b_0, b_1)))) = d((a_0, a_1), (a_0, 0)) \\
\leq \eta.
\]

Thus \(\bar{F}_0, \bar{F}_1\) and \(\bar{G}\) satisfy \(\bar{F}_0(\bar{G}(x_0, x_1)) \approx x_0\) within \(\eta\) on \((B, d)\).
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Thus \(\overline{F}_0, \overline{F}_1\) and \(\overline{G}\) satisfy \(\overline{F}_0(\overline{G}(x_0, x_1)) \approx x_0\) within \(\eta\) on \((B, d)\). The approximate satisfaction of \(\overline{F}_1(\overline{G}(x_0, x_1)) \approx x_1\) is handled similarly. Thus

\[
\lambda((B, d), \Sigma) \leq \eta.
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Precise definitions for discontinuous satisfaction. \( \mu_n(A, \Sigma) \).

An example: \( \mu_1(A, \Sigma) = 0; \mu_2(A, \Sigma) = \text{diam}(A) \).

Some further results

Algorithmic questions about \( \lambda \) and \( \mu_n \).
Let \((A,d),(B,e)\) be metric spaces. Given \(F:B \to A\) (not necessarily continuous) and \(\delta, \varepsilon > 0\), we say that \(F\) is \((\delta, \varepsilon)\)-constrained if it satisfies: for all \(b, b' \in B\), if \(e(b, b') < \delta\), then \(d(F(b), F(b')) < \varepsilon\).

(Uniform continuity rephrased: for every \(\varepsilon > 0\) there exists \(\delta > 0\) so that \(F\) is constrained by \((\delta, \varepsilon)\).)

We say that \(F\) is \(n\)-constrained by \((\delta_0, \delta_n)\) iff there exist \(0 < \delta_0 \leq \delta_1 \leq \cdots \leq \delta_n\) such that \(F\) is \((\delta_0, \delta_1)\)-constrained and \((\delta_1, \delta_2)\)-constrained, and so on, up to \((\delta_{n-1}, \delta_n)\)-constrained.
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We say that \( F \) is \( n \)-constrained by (\( δ_0, δ_n \)) iff there exist \( 0 < δ_0 \leq δ_1 \leq \cdots \leq δ_n \) such that \( F \) is (\( δ_0, δ_1 \))-constrained and (\( δ_1, δ_2 \))-constrained, and so on, up to (\( δ_{n-1}, δ_n \))-constrained.
Lemma
Suppose that $f$ maps a convex subset of $\mathbb{R}$ into $\mathbb{R}$, and that $f$ is $(\delta, \varepsilon)$-constrained with $\delta, \varepsilon > 0$. If $a < c$ and $s$ is between $f(a)$ and $f(c)$, then there exists $b$ with $a \leq b \leq c$ and with $d(f(b), s) < \varepsilon/2$. 
Definitions of $\models^\varepsilon_n$ and $\mu_n(A, \Sigma)$

$A \models^\varepsilon_n \Sigma$
Definitions of $\models_n^\varepsilon$ and $\mu_n(A, \Sigma)$

$A \models_n^\varepsilon \Sigma$

means that there exists an algebra $A = (A, \overline{F}_t)_{t \in T}$ modeling $\Sigma$ and a real number $\delta_0 > 0$ such that each $\overline{F}_t$ is $n$-constrained by $(\delta_0, \varepsilon)$.
Definitions of $\models_n^\varepsilon$ and $\mu_n(A, \Sigma)$

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means that there exists an algebra $A = (A, \overline{F}_t)_{t \in T}$ modeling $\Sigma$ and a real number $\delta_0 > 0$ such that each $\overline{F}_t$ is $n$-constrained by $(\delta_0, \varepsilon)$.

We define

$$\mu_n(A, \Sigma) = \inf \{ \varepsilon : A \models_n^\varepsilon \Sigma \}. $$
Definitions of $\models_{\varepsilon}^{n}$ and $\mu_{n}(A, \Sigma)$

$A \models_{\varepsilon}^{n} \Sigma$

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We define

$$\mu_{n}(A, \Sigma) = \inf \{ \varepsilon : A \models_{\varepsilon}^{n} \Sigma \}.$$ 

It is not hard to see that

$$0 \leq \mu_{1}(A, \Sigma) \leq \mu_{2}(A, \Sigma) \leq \cdots \leq \text{diam}(A).$$
We repeat the definition:

\[ A \models^\varepsilon_n \Sigma \]

means that there exists an algebra \( A = (A, \overline{F}_t)_{t \in T} \) modeling \( \Sigma \) and a real number \( \delta_0 > 0 \) such that each \( \overline{F}_t \) is \( n \)-constrained by \( (\delta_0, \varepsilon) \).
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Thus if \( A \models \Sigma \), then \( A \models^n_{\varepsilon} \Sigma \) for every \( n \) and every \( \varepsilon > 0 \),
Connection of $\models^n_\varepsilon$ and $\mu_n(A, \Sigma)$ with $A \models \Sigma$

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Thus if $A \models \Sigma$, then $A \models^n_\varepsilon \Sigma$ for every $n$ and every $\varepsilon > 0$, and hence

$$\mu_n(A, \Sigma) = \inf \{ \varepsilon : A \models^n_\varepsilon \Sigma \} = 0$$

for every $n$. 
Six sections of the talk.

Basics

Some calculations on approximate satisfaction. $\lambda(A, \Sigma)$.

Precise definitions for discontinuous satisfaction. $\mu_n(A, \Sigma)$.

An example: $\mu_1(A, \Sigma) = 0; \mu_2(A, \Sigma) = \text{diam}(A)$.

Some further results

Algorithmic questions about $\lambda$ and $\mu_n$. 
Our first $\Sigma$ is taken as before: succinctly, it says that $A_2 \xrightarrow{G} A_F \xrightarrow{} A_2 = \text{identity}$. So let $F$ be a Peano curve: continuous from $[0,1]$ onto $[0,1]$, and let $G$ be any left-inverse to $F$. ($G$ is perforce discontinuous.) For arbitrary $\varepsilon > 0$, define functions $F'$ and $G'$ via $G'(a_0,b_0) = \varepsilon G(a_0,b_0)$; $F'(a) = F(1 \wedge (a/\varepsilon))$. Now the discontinuities of $G'$ are no larger than $\varepsilon$, and $F'$ remains continuous, while $F'$ and $G'$ still satisfy $\Sigma$. Thus $A_1 = \varepsilon 1$ for every $\varepsilon > 0$; hence $\mu_1([0,1], \Sigma) = 0$. 

$\mu_1([0,1], \Sigma) = 0$ for $\Sigma = \text{injective binary}$
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So let $\bar{F}$ be a Peano curve: continuous from $[0, 1]$ \textbf{onto} $[0, 1]^2$, and let $\bar{G}$ be any left-inverse to $\bar{F}$.
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So let \( \overline{F} \) be a Peano curve: continuous from \([0, 1]\) onto \([0, 1]^2\), and let \( \overline{G} \) be any left-inverse to \( \overline{F} \). (\( \overline{G} \) is perforce discontinuous.) For arbitrary \( \varepsilon > 0 \), define functions \( F' \) and \( G' \) via

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Now the discontinuities of \( G' \) are no larger than \( \varepsilon \), and \( F' \) remains continuous, while \( F' \) and \( G' \) still satisfy \( \Sigma \). Thus \( A \models_{\varepsilon} \) for every \( \varepsilon > 0 \); hence \( \mu_1([0, 1], \Sigma) = 0 \). \[\square\]
\[ \mu_2([0, 1], \Sigma) = 1 \]
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We consider \((A; \overline{G}, \overline{F}_0, \overline{F}_1)\) modeling \(\Sigma\), with the operations \((\delta_0, \delta_1)\)-constrained and \((\delta_1, \delta_2)\)-constrained. We will show that \(\delta_2 \geq 1\).
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Let \(\{a_0, a_1\} = \{b_0, b_1\} = \{0, 1\}\). Compare the four values \(\bar{G}(a_i, b_j)\); w.l.o.g. \(\bar{G}(a_0, b_0)\) is the smallest.
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We consider the real function \(\overline{H}(x) = \overline{G}(a_0, x)\). By the Lemma (IVT), there exists \(e \in [0, 1]\) with

\[
d(\overline{G}(a_0, e), \overline{G}(a_1, b_0)) < \delta_1.
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$\mu_2([0, 1], \Sigma) = 1$, concluded

Repeat: $d(\overline{G}(a_0, e), \overline{G}(a_1, b_0)) < \delta_1$. 

Thus $\mu_2([0, 1], \Sigma)$, being the infimum of such $\delta_2$'s, must be 1.
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Repeat: $d(\overline{G}(a_0, e), \overline{G}(a_1, b_0)) < \delta_1$.

Because $\{a_0, a_1\} = \{0, 1\}$, because of $\Sigma$, and because the function $\overline{F}_0$ is $(\delta_1, \delta_2)$-constrained, we now have:

$$1 = d(a_0, a_1) = d(\overline{F}_0(\overline{G}(a_0, e)), \overline{F}_0(\overline{G}(a_1, b_0))) \leq \delta_2.$$
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Basics

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Some further results

Algorithmic questions about $\lambda$ and $\mu_n$. 
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- $\mu_2([0,1]^n, \text{Groups}) = \text{diameter}([0,1]^n)$.
- $\mu_3(Y, \text{Lattices}) \geq 0.5$.

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- $\mu_1(S^1, \Sigma) = \frac{2}{3}$.
- $\mu_1(S^k, \Sigma) \geq \frac{2}{3}$.

We will sketch the proof of this last result...
\( \mu_1(S^k, \Sigma) \geq 2/3; \quad \Sigma = \text{ternary majority.} \)
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We show that if \( F \) is a majority operation on \( S^k \) that is \((\delta, 2/3)\)-constrained for some \( \delta > 0 \), then \( F \) can be deformed into a continuous majority operation \( G \). This would contradict the fact that \( S^k \not\models \Sigma \).
\( \mu_1(S^k, \Sigma) \geq 2/3; \quad \Sigma = \text{ternary majority.} \)

We show that if \( \overline{F} \) is a majority operation on \( S^k \) that is \((\delta, 2/3)\)-constrained for some \( \delta > 0 \), then \( \overline{F} \) can be deformed into a continuous majority operation \( \overline{G} \). This would contradict the fact that \( S^k \not\models \Sigma \). The non-existence of such a constrained operation is tantamount to the inequality \( \mu_1(S^k, \Sigma) \geq 2/3 \).
\[ \mu_1(S^k, \Sigma) \geq 2/3; \quad \Sigma = \text{ternary majority}. \]

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**Lemma**

*If \( F \) is a finite subset of \( S^k \) with \( \text{diameter}(F) < 2/3 \), then there is a convex subset \( A \) of \( S^k \) such that \( F \subseteq A \).*
Recall that $\overline{F}$ is a majority operation on $S^k$ that is $(\delta, 2/3)$-constrained.
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We begin with a triangulation of $(S^k)^3$ that contains sub-complexes triangulating $\{(x, x, z) : x, z \in S^k\}$ and so on, and such that each simplex has diameter $< \delta$. 

$\mu_1(S^k, \text{Ternary majority}) \geq 2/3$, continued
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By the lemma, $\overline{F}$ maps each simplex into a convex set $K$, which is a universal extensor in topology; i.e. maps into $K$ can be extended continuously to larger sets. The desired map $\overline{G}$ is then constructed one simplex at a time, by recursion on the simplex dimension.
Six sections of the talk.

Basics

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An example: $\mu_1(A, \Sigma) = 0; \mu_2(A, \Sigma) = \text{diam}(A)$.

Some further results

Algorithmic questions about $\lambda$ and $\mu_n$. 
Intractability of an algorithmic approach to compatibility.
Recall that $\mathbb{R} \models \Sigma$ is not a recursive (algorithmic) property of finite $\Sigma$. 

In fact there is no space $A$ for which $A \models \Sigma$ is known to be algorithmic (besides a few known spaces $A$ where $A \models \Sigma$ iff $\Sigma$ is essentially trivial). E.g. it remains unknown whether $[0,1] \models \Sigma$ is algorithmic.
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... Looking at $\lambda$, we gain traction    →
Recursive enumeration of approximate satisfaction.

We fix a list of operation symbols $F_i \ (i \in \omega)$, which includes each arity infinitely often. Clearly every finite set of equations is definitionally equivalent to a finite set involving only the operation symbols $F_i$. 
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**Theorem**

Let $K$ be a finite simplicial complex, with $|K|$ its geometric realization (as a topological space), and let $\alpha > 0$ be a computable real number. There is an algorithm $A_{K,\alpha}$ whose output consists of those finite sets $\Sigma$ of equations in $F_i \ (i \in \omega)$ for which $\lambda(|K|, \Sigma) < \alpha$. 
Informal description of the algorithm $A_{K,\alpha}$.

Fix for the moment a single finite $\Sigma$, for simplicity having a single operation $F_0$, which is $p$-ary, while also fixing $M, N \in \omega^+$. We consider the complexes $C = (K^p)^{(M)}$ and $D = K^{(N)}$, where $(M)$ denotes the $M$-th subdivided complex. And let us consider the case where $\alpha$ is rational.
Informal description of the algorithm $A_{K,\alpha}$.

Fix for the moment a single finite $\Sigma$, for simplicity having a single operation $F_0$, which is $p$-ary, while also fixing $M, N \in \omega^+$. We consider the complexes $C = (K^p)^{(M)}$ and $D = K^{(N)}$, where $(M)$ denotes the $M$-th subdivided complex. And let us consider the case where $\alpha$ is rational.

For each simplicial map $C \rightarrow D$ (of which there are finitely many), we may check whether the corresponding continuous map, let us call it $\overline{F}_0: |K|^p \rightarrow K$, satisfies $\Sigma$ within $< \alpha$. In fact this proposition lies in the first-order theory of reals (using barycentric co-ordinates), and so Tarski’s algorithm will yield an algorithm to check this.
In case we discover a simplicial map satisfying $\Sigma$ within $\alpha$, we allow this fragment of the algorithm $A_{K,\alpha}$ to output $\Sigma$. 
In case we discover a simplicial map satisfying $\Sigma$ within $\alpha$, we allow this fragment of the algorithm $A_{K,\alpha}$ to output $\Sigma$. The Simplicial Approximation Theorem ultimately says that if $A \models _{\beta} \Sigma$ for some $\beta < \alpha$, then one of these simplicial maps will in fact satisfy $\Sigma$ within $\alpha$. Therefore if we loop the above subroutine through all $M$, all $N$, and all appropriate simplicial maps, we will sooner or later output $\Sigma$ if and only if $A \models _{\alpha} \Sigma$. All that remains is to consider all $\Sigma$. By a suitable interlacing, we may run the above process concurrently for all of them, thereby yielding an algorithm that outputs exactly those $\Sigma$ with $A \models _{\alpha} \Sigma$. This completes our description of $A_{K,\alpha}$. 
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All that remains is to consider all $\Sigma$. By a suitable interlacing, we may run the above process concurrently for all of them, thereby yielding an algorithm that outputs exactly those $\Sigma$ with $A \models_\alpha \Sigma$. This completes our description of $A_{K,\alpha}$. 
An algorithm related to $\lambda(|K|, \Sigma) = 0$. 

**Corollary**

There is an algorithm $F$ that takes no input, and whose output is an infinite stream of triples $(K, \Sigma, s)$, with each $K$ a finite complex, with each $\Sigma$ a finite set of equations in the symbols $F_i$ ($i \in \omega$), and with each $s \in \mathbb{Z}^+$; such that the following condition holds: arbitrary $K$ and $\Sigma$ satisfy $\lambda(|K|, \Sigma) = 0$ iff $(K, \Sigma, s)$ occurs in the output stream of $F$ for arbitrarily large $s$. 

**Proof.** Interlace the algorithms $A_{K, 1/s}$ for $s = 1, 2, 3, \ldots$, and for all finite $K$. When $A_{K, 1/s}$ outputs $\Sigma$, then $F$ will output $(K, \Sigma, s)$. 

An algorithm related to \( \lambda(|K|, \Sigma) = 0 \).

Corollary

There is an algorithm \( \mathcal{F} \) that takes no input, and whose output is an infinite stream of triples \((K, \Sigma, s)\), with each \( K \) a finite complex, with each \( \Sigma \) a finite set of equations in the symbols \( F_i \) \((i \in \omega)\), and with each \( s \in \mathbb{Z}^+ \); such that the following condition holds: arbitrary \( K \) and \( \Sigma \) satisfy \( \lambda(|K|, \Sigma) = 0 \) iff \((K, \Sigma, s)\) occurs in the output stream of \( \mathcal{F}_K \) for arbitrarily large \( s \).

Proof.

Interlace the algorithms \( \mathcal{A}_{K,1/s} \) for \( s = 1, 2, 3, \cdots \), and for all finite \( K \). When \( \mathcal{A}_{K,1/s} \) outputs \( \Sigma \), then \( \mathcal{F} \) will output \((K, \Sigma, s)\). \( \square \)
Arithmetical character of $K \models \Sigma$, $\lambda(|K|, \Sigma) = 0$ and $\mu_n(|K|, \Sigma) = 0$.

The above corollary is tantamount to saying that the set of all $\Sigma$ with $\lambda(|K|, \Sigma) = 0$ is $\Pi^0_2$ in the arithmetical hierarchy.
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The above corollary is tantamount to saying that the set of all $\Sigma$ with $\lambda(|K|, \Sigma) = 0$ is $\Pi^0_2$ in the arithmetical hierarchy.

Note that we do not have such a result for $|K| \models \Sigma$. Nor do we know how to obtain such a result for $\mu_n(|K|, \Sigma) = 0$. The problem here would be that the validity of $|K| \models^e_n \Sigma$ requires some operations on $|K|$ that exactly model $\Sigma$. And clearly simplicial maps will not generally satisfy $\Sigma$ exactly. (Example: groups on $S^1$).