The model companion of the class of pseudo-complemented semilattices is finitely axiomatizable
(joint work with Regula Rupp and Jürg Schmid)

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Point of departure

Axiomatizing $PCS^{ac}$
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Point of departure

Axiomatizing $PCS^{ac}$

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Basic notions

Algebraic and existential closedness
**Basic notions**

**Algebraic and existential closedness**

Let $K$ be a class of (universal) algebras of the same (finitary) type, and $L_K$ a (first-order) language suitable for $K$. For $A \in K$, let $L_{K,A}$ be $L$ extended with an individual constant for each element $a \in A$. 

**Definition**

(i) An algebra $A \in K$ is called **algebraically closed** - for short: ac - iff for every finite set $\phi_1, \ldots, \phi_n$ of positive $\exists$-sentences from $L_{K,A}$, the following holds: Whenever $\bigwedge n \phi_i$ is satisfied in some extension $A \subseteq A' \in K$, then it is satisfied already in $A$.

(ii) $A \in K$ is called **existentially closed** - for short: ec - iff (the same as sub (i), with "positive" removed).
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Definition: (i) An algebra $A \in K$ is called algebraically closed in $K$ - for short: ac - iff for every finite set $\phi_1, \ldots, \phi_n$ of positive $\exists_1$-sentences from $\mathcal{L}_{K,A}$ the following holds: Whenever $\bigwedge_1^n \phi_i$ is satisfied in some extension $A \subseteq A' \in K$, then it is satisfied already in $A$.

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Basic notions

Algebraic and existential closedness for the working mathematician

$A$ is ac in $K$ iff every finite set of equations with parameters from $A$ which is solvable in some extension $A \subseteq A' \in K$ already has a solution in $A$, and it is ec in $K$ iff the same holds for every such finite set of equations and negated equations.
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Explicitly, we are dealing with finite sets of formulae of type $s(x_1, \ldots, x_m, a_1, \ldots, a_n) = t(x_1, \ldots, x_m, a_1, \ldots, a_n)$ resp. $s(x_1, \ldots, x_m, a_1, \ldots, a_n) \neq t(x_1, \ldots, x_m, a_1, \ldots, a_n)$, where $s$ and $t$ are $L$-terms and $a_1, \ldots, a_n \in A$. 
Basic notions

Algebraic versus existential closedness

If $K$ is a field and $p(\rightarrow x)$ and $q(\rightarrow x)$ are polynomials over $K$, then the satisfiability of the inequality $p(\rightarrow x) \neq q(\rightarrow x)$ is equivalent to the satisfiability of the equation $x \cdot (p(\rightarrow x) - q(\rightarrow x)) = 1$, where $\rightarrow x = (x_1, \ldots, x_n)$.

This is not the general situation: In the class of boolean algebras every boolean algebra is algebraically closed whereas boolean algebra is existentially closed if and only if it is atomfree. An abelian group is algebraically closed if and only if it is divisible, whereas it is existentially closed if and only if it is divisible and contains an infinite direct sum of copies of $\mathbb{Q}/\mathbb{Z}$ (as a module).
**Basic notions**

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The two notions can coincide as in the class of fields:

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Basic notions

The classes $K^{ac}$ and $K^{ec}$
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If a class of algebras $K$ is (finitely) axiomatizable (with first-order sentences) the natural question arises whether the subclasses $K^{ac}$ and $K^{ec}$ are (finitely) axiomatizable.
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A semantic characterization of finite axiomatizability of a class of $\mathcal{L}$-structures:
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If a class of algebras $K$ is (finitely) axiomatizable (with first-order sentences) the natural question arises whether the subclasses $K^{ac}$ and $K^{ec}$ are (finitely) axiomatizable.

A semantic characterization of finite axiomatizability of a class of $L$-structures:

**Theorem:** An finitely axiomatizable class of $L$-structures is *finitely axiomatizable* iff both the class itself as well as its complementary class are closed under elementary equivalence and ultraproducts.
Basic notions

The classes $K^{ac}$ and $K^{ec}$

- If $BA$ is the class of boolean algebras, then adding the sentence $(\forall x)(\exists y)(0 < x \rightarrow 0 < y < x)$ to the axioms of boolean algebras yields a finite axiomatization of $BA^{ec}$.
Basic notions

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- If $BA$ is the class of boolean algebras, then adding the sentence $(\forall x)(\exists y)(0 < x \rightarrow 0 < y < x)$ to the axioms of boolean algebras yields a finite axiomatization of $BA^{ec}$.

- If $GA$ is the class of abelian groups, then adding the infinite set of sentence $\{(\forall x)(\exists y)(x = ny) \mid n \in \mathbb{N}\}$ to the axioms of abelian groups yields an infinite axiomatization of $GA^{ac}$. There is no finite axiomatization of $GA^{ac}$.
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- Let $\mathcal{L}$ be the first-order language without relation, function and constant symbols and $K$ be the class of all $\mathcal{L}$-models (of the empty theory). Then $K$ is simply the class of sets. We obtain $K^{ac} = K$ and $K^{ec} = \{x \in K \mid x$ is infinite\}. Thus $K^{ac}$ is finitely axiomatizable and $K^{ec}$ is not.
Basic notions

The model companion of a class of $\mathcal{L}$-structures

We will not discuss this model theoretic notion here. For our purposes it suffices to notice that under very general assumptions on a class $\mathcal{K}$ of algebras the model companion of $\mathcal{K}$ coincides with $\mathcal{K}_{ec}$.

The class $\mathcal{PCS}$ of pseudo-complemented semilattices satisfies these assumptions.
Basic notions

The model companion of a class of $\mathcal{L}$-structures

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Basic notions

The model companion of a class of \( \mathcal{L} \)-structures

We will not discuss this model theoretic notion here. For our purposes it suffices to notice that under very general assumptions on a class \( K \) of algebras the model companion of \( K \) coincides with \( K^{ec} \).

The class \( PCS \) of pseudo-complemented semilattices satisfies these assumptions.
Basic notions

Pseudo-complemented semilattices: definition

Definition: A pseudo-complemented semilattice (for short: p-semilattice) \((P; \land, *, 0, 1)\) is a meet-semilattice \((P; \land)\) with least element 0 and top element 1, equipped with an unary operation \(a \mapsto \overline{a}\) such that for all \(x \in P\), \(x \land \overline{a} = 0\) iff \(x \leq \overline{a}\).

With PCS we denote the class of all p-semilattices.

Theorem (Frink 1962) PCS is a variety.

Definition: A distributive pseudo-complemented lattice (for short: p-algebra) \((L; \lor, \land, *, 0, 1)\) is a distributive lattice \((L; \lor, \land)\) with least element 0 and top element 1, equipped with an unary operation \(a \mapsto \overline{a}\) such that for all \(x \in L\), \(x \land \overline{a} = 0\) iff \(x \leq \overline{a}\). Let PALG be the class of all p-algebras.
Basic notions

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**Definition:** A pseudo-complemented semilattice (for short: p-semilattice) \((P; \wedge, *, 0, 1)\) is a meet-semilattice \((P; \wedge)\) with least element 0 and top element 1, equipped with an unary operation \(a \mapsto a^*\) such that for all \(x \in P\), \(x \wedge a = 0\) iff \(x \leq a^*\). With \(PCS\) we denote the class of all p-semilattices.

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**Basic notions**

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Basic notions

Pseudo-complemented semilattices: examples
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Pseudo-complemented semilattices: examples

- Every boolean algebra \( B \) considered as a \((B; \land, *, 0, 1)\)-reduct is a pcs. \( 2, F_n \) and \( A \) denote the two-element boolean algebra, the \( n \)-atom boolean algebra and the countable atomless boolean algebra, respectively.
Basic notions

Pseudo-complemented semilattices: examples

- Every boolean algebra $B$ considered as a $(B; \land, *, 0, 1)$-reduct is a pcs. $\mathbf{2}$, $\mathbf{F}_n$ and $\mathbf{A}$ denote the two-element boolean algebra, the $n$-atom boolean algebra and the countable atomless boolean algebra, respectively.

- For an arbitrary boolean algebra $B$ the ordered structure $\hat{B}$ obtained from $B$ by adding a new top element is a pcs. $\mathbf{3}$ denotes three element chain which is obtained from $\mathbf{2}$ by adding a new top element.
Basic notions

Pseudo-complemented semilattices: special elements
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Pseudo-complemented semilattices: special elements

Let $x \in A \in PCS$.

- $x$ is dense iff $x^* = 0$. $D(A) = \{x \in A : x$ dense$\}$
  = the dense filter of $A$.  

Basic notions

Pseudo-complemented semilattices: special elements

Let $x \in A \in PCS$.

- $x$ is dense iff $x^* = 0$. $D(A) = \{x \in A : x \text{ dense}\}$ = the dense filter of $A$.
- $x$ is skeletal iff $x^{**} = x$. $Sk(A) = \{x \in A : x \text{ skeletal}\}$ = the skeleton of $A$. 
Basic notions

Pseudo-complemented semilattices: some facts
Basic notions

Pseudo-complemented semilattices: some facts

- If $P = \langle P; \land, *, 0, 1 \rangle$ is a pcs, then $\langle \text{Sk}(P); \land, \lor, *, 0, 1 \rangle$ putting $a \lor b := (a^* \land b^*)^*$ is a boolean algebra.
Basic notions

Pseudo-complemented semilattices: some facts

- If \( P = \langle P; \land, \ast, 0, 1 \rangle \) is a pcs, then \( \langle \text{Sk}(P); \land, \lor, \ast, 0, 1 \rangle \) putting \( a \lor b := (a^\ast \land b^\ast)^\ast \) is a boolean algebra.

- \( \hat{B} \), where \( B \) is any boolean algebra, are exactly the subdirectly irreducible members of \( PCS \).
Basic notions

Pseudo-complemented semilattices: some facts

- If $P = \langle P; \land, *, 0, 1 \rangle$ is a pcs, then $\langle \text{Sk}(P); \land, \lor, *, 0, 1 \rangle$ putting $a \lor b := (a^* \land b^*)^*$ is a boolean algebra.

- $\hat{\mathcal{B}}$, where $\mathcal{B}$ is any boolean algebra, are exactly the subdirectly irreducible members of $PCS$.

- The three-element chain $3$ generates the variety $PCS$, that is $PCS = \text{HSP}(3)$. The subclass $BA = \text{HSP}(2)$ of boolean algebras is the only non-trivial subvariety.
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In their 1986 paper *Finite axiomatizations for existentially closed posets and semilattices*, Order *3*(1986), 169–178, Michael Albert and Stanley Burris asked the following question:

“Does the class of pseudo-complemented semilattices have a finitely axiomatizable model companion?”
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which is the same as asking

“Is $PCS^{ec}$ finitely axiomatizable?”

In the above paper the authors answered the same question for the class of *posets* and the class of *semilattices* in the affirmative.
Point of departure

For the following classes $K$ of algebras with a pseudo-complement the above question was answered in the affirmative:
Point of departure

For the following classes $K$ of algebras with a pseudo-complement the above question was answered in the affirmative:

- $K$ the class of pseudo-complemented distributive lattices (J. Schmid, 1982).
  The same author showed that this also holds for all Lee classes $B_n$ which are subvarieties of $K$. 

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[$K$]: The class of pseudo-complemented distributive lattices.
[$B_n$]: Lee classes.
[PCS]: Class of pseudo-complemented semilattices.
[ISP]: Class of implicative pseudo-complemented semilattices.
[Schmid]: J. Schmid.
[1982]: Year of publication.
[Gerber]: Gerber.
[1991]: Year of publication.

For the following classes $K$ of algebras with a pseudo-complement the above question was answered in the affirmative:

- $K$ the class of pseudo-complemented distributive lattices (J. Schmid, 1982). The same author showed that this also holds for all Lee classes $B_n$ which are subvarieties of $K$.

- $K$ the class of the so-called Stone semilattices alias the class $\mathbb{ISP}(3) \subseteq PCS$, which is a quasi-subvariety of $PCS$ (Gerber, Schmid, 1991).
A finite axiomatization of $PCS_{ac}$

A semantic characterization of $PCS_{ac}$
A finite axiomatization of $PCS^{ac}$

A semantic characterization of $PCS^{ac}$

The following theorem is crucial for finding a finite axiomatization of $PCS^{ac}$ as well as $PCS^{ec}$.

(J. Schmid, 1985) A p-semilattice $P$ is algebraically closed iff for any finite subalgebra $F \leq P$ there exists $r, s \in \mathbb{N}$ and a p-semilattice $F'$ isomorphic to $2^r \times \hat{A}^s$ such that $F \leq F' \leq P$. 
A finite axiomatization of $P\mathcal{CS}^{ac}$

The list of axioms
A finite axiomatization of $PCS^{ac}$

The list of axioms

- (AC1) $(\forall a, b, c \in P)(\exists x, y \in P)(c \geq a \land b \rightarrow x \geq a, y \geq b \land x \land y = c)$
A finite axiomatization of $PCS^{ac}$

The list of axioms

- **(AC1)** $(\forall a, b, c \in P)(\exists x, y \in P)(c \geq a \land b \rightarrow x \geq a, \ y \geq b \land x \land y = c)$

- **(AC2)** $(\forall a, b, c, t \in P)(\exists x \in P)(a^* = b^* = c^* = 0, \ c < b < a, \ t \land c < t \land b < t \land a \rightarrow c < x < a, \ x \land b = c, \ t \land c < t \land x < t \land a)$
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- **(AC3)** $(\forall d, d_m \in D(P), f, f_m, x \in P, k \in Sk(P))(\exists z_x \in Sk(P))(d \parallel d_m, \ f \leq d_m, \ f_m \leq d, \ f_m \not\leq d_m, \ k \leq d, \ k^* \land f \not\leq d, \ x^* \leq d_m \rightarrow k \leq z_x \leq d, \ z_x^* \land f \not\leq d, \ z_x \land f_m \not\leq d_m \land (z_x \land x)^* \leq d_m)$. 
A finite axiomatization of $PCS^{ac}$

The list of axioms

- (AC1) $(\forall a, b, c \in P)(\exists x, y \in P)(c \geq a \land b \rightarrow x \geq a, y \geq b \land x \land y = c)$

- (AC2) $(\forall a, b, c, t \in P)(\exists x \in P)(a^* = b^* = c^* = 0, c < b < a, t \land c < t \land b < t \land a \rightarrow c < x < a, x \land b = c, t \land c < t \land x < t \land a)$

- (AC3) $(\forall d, d_m \in D(P), f, f_m, x \in P, k \in Sk(P))(\exists z_x \in Sk(P))(d \parallel d_m, f \leq d_m, f_m \leq d, f_m \not< d_m, k \leq d, k^* \land f \not< d, x^* \leq d_m \rightarrow k \leq z_x \leq d, z_x^* \land f \not< d, z_x \land f_m \not< d_m \land (z_x \land x)^* \leq d_m).$

- (AC4) $(\forall d \in D(P), b_1 \in Sk(P)(\exists b_2 \in Sk(P))(b_1 < d < 1 \rightarrow b_1 < b_2 < d \land b_1 \lor b_2^* < d)$
A finite axiomatization of $\text{PCS}^{ac}$

The theorem
A finite axiomatization of $PCS^{ac}$

The theorem

Finally, a finite axiomatization of $PCS^{ac}$:

(−, R. Rupp, J. Schmid, 2012) Let $\mathcal{A}$ be a finite list of first-order sentences axiomatizing $PCS$. Then the finite set $\mathcal{A} \cup \{(AC1), (AC2), (AC3), (AC4)\}$ axiomatizes $PCS^{ac}$. 
A finite axiomatization of $PCS_{\text{ac}}$

Proof sketch
A finite axiomatization of $PCS^{ac}$

Proof sketch

Let $P$ be a p-semilattice satisfying (AC1), (AC2), (AC3) and (AC4) and $F_0 \leq P$, $|F_0| < \omega$. By repeatedly applying axiom $(ACi)$, $i = 1, \ldots, 4$, we construct a chain $F_0 \leq F_1 \leq F_2 \leq F_3 \leq F_4 \leq P$ s.t.
A finite axiomatization of \( \text{PCS}^{ac} \)

Proof sketch

Let \( P \) be a \( p \)-semilattice satisfying (AC1), (AC2), (AC3) and (AC4) and \( F_0 \leq P, |F_0| < \omega \). By repeatedly applying axiom (AC\( i \)), \( i = 1, \ldots, 4 \), we construct a chain \( F_0 \leq F_1 \leq F_2 \leq F_3 \leq F_4 \leq P \) s.t.

- \( F_1 \) is finite and distributive,
A finite axiomatization of $PCS^{ac}$

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- $F_1$ is finite and distributive,

- $F_2$ is finite, distributive and there is $s \in \mathbb{N}$ s.t. $D(F_2) \cong \{e, 1\}^s$, where $D(F_2) = \{e, 1\}$ and $\cong$ denotes isomorphism.
Proof sketch

Let $P$ be a p-semilattice satisfying (AC1), (AC2), (AC3) and (AC4) and $F_0 \leq P$, $|F_0| < \omega$. By repeatedly applying axiom (AC$i$), $i = 1, \ldots, 4$, we construct a chain $F_0 \leq F_1 \leq F_2 \leq F_3 \leq F_4 \leq P$ s.t.

- $F_1$ is finite and distributive,

- $F_2$ is finite, distributive and there is $s \in \mathbb{N}$ s.t. $D(F_2) \cong \{e, 1\}^s$,

- there is $r \in \mathbb{N}$ s.t. $F_3 \cong 2^r \times \prod_{i=1}^{s} \hat{F}_{f(i)}$, $f(i) \in \mathbb{N}$, $i = 1, \ldots, s$,
A finite axiomatization of $PCS^{ac}$

Proof sketch

Let $P$ be a p-semilattice satisfying (AC1), (AC2), (AC3) and (AC4) and $F_0 \leq P$, $|F_0| < \omega$. By repeatedly applying axiom (AC$i$), $i = 1, \ldots, 4$, we construct a chain $F_0 \leq F_1 \leq F_2 \leq F_3 \leq F_4 \leq P$ s.t.

- $F_1$ is finite and distributive,

- $F_2$ is finite, distributive and there is $s \in \mathbb{N}$ s.t. $D(F_2) \cong \{e, 1\}^s$,

- there is $r \in \mathbb{N}$ s.t. $F_3 \cong 2^r \times \prod_{i=1}^{s} \widehat{F_{f(i)}}$, $f(i) \in \mathbb{N}$, $i = 1, \ldots, s$,

- $F_4 \cong 2^r \times \left(\widehat{\mathbb{A}}\right)^s$. 
A finite axiomatization of $PCS^{ec}$

Existential closedness revisited
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Every finite system of equations and inequalities with coefficients $a_1, \ldots, a_m \in P$ corresponds to a quantifier-free formula $\varphi(\vec{x}, \vec{a})$. This yields:
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- $P$ is ec iff for any extension $Q$ of $P$ and any quantifier-free formula $\varphi$ from $\mathcal{L}(P)$ the following holds: If $Q \models (\exists x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n, a_1, \ldots, a_m)$ then there are $u_1, \ldots, u_n \in P$ such that $P \models \varphi(\vec{u}, \vec{a})$.
Existential closedness revisited

Every finite system of equations and inequalities with coefficients $a_1, \ldots, a_m \in P$ corresponds to a quantifier-free formula $\varphi(x^d, \bar{a})$. This yields:

- $P$ is ec iff for any extension $Q$ of $P$ and any quantifier-free formula $\varphi$ from $\mathcal{L}(P)$ the following holds: If $Q \models (\exists x_1, \ldots, x_n) \varphi(x_1, \ldots, x_n, a_1, \ldots, a_m)$ then there are $u_1, \ldots, u_n \in P$ such that $P \models \varphi(\overrightarrow{u}, \bar{a})$.

- $P$ is ec iff for any extension $Q$ of $P$, arbitrary $S := \{a_1, \ldots, a_m\} \subset P$ and $V := \{v_1, \ldots, v_n\} \subset Q$ there is $U := \{u_1, \ldots, u_n\} \subset P$ and an isomorphism $f : S \cup U \to S \cup V$ with $f|_S = \text{id}_S$. 
A finite axiomatization of $PCS^{ee}$

Assumptions on the set of coefficients and the solution set
A finite axiomatization of $PCS^{ec}$

Assumptions on the set of coefficients and the solution set $S \subset P$ and the set $T := S \cup V \subset Q$ the following can be assumed without loss of generality:
A finite axiomatization of $PCS^{\text{sec}}$

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$S \subset P$ and the set $T := S \cup V \subset Q$ the following can be assumed without loss of generality:

$$S \cong \hat{F}_t \quad (t \in \mathbb{N})$$

with $\hat{F}_0 := 2$ and
A finite axiomatization of $\text{PCS}^{ec}$

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with $\widehat{F}_0 := 2$ and

$T \cong 2^p \times \widehat{F}_u^q \ (p, q, u \in \mathbb{N})$
A finite axiomatization of $PCS^{ee}$

The list of axioms
A finite axiomatization of $PCS^{ee}$

The list of axioms
A finite axiomatization of $PCS_{ec}$

The list of axioms

- (EC1) $(\forall b_1, b_2)(\exists b_3)((Sk(b_1) \& Sk(b_2) \& b_1 < b_2) \rightarrow (Sk(b_3) \& b_1 < b_3 < b_2))$
A finite axiomatization of $PCS^{ec}$

The list of axioms

- (EC1) $(\forall b_1, b_2)(\exists b_3)((Sk(b_1) \& Sk(b_2) \& b_1 < b_2) \rightarrow (Sk(b_3) \& b_1 < b_3 < b_2))$

- (EC2) $(\forall b_1, d)(\exists b_2)((Sk(b_1) \& D(d) \& b_1 < d \& b_1^* \parallel d) \rightarrow (Sk(b_2) \& b_1 < b_2 \parallel d \& b_2 < 1 \& b_1 \lor b_2^* < d \& b_1^* \land b_2 \parallel d))$
The list of axioms

- **(EC1)** \((\forall b_1, b_2)(\exists b_3)((Sk(b_1) \land Sk(b_2) \land b_1 < b_2) \rightarrow (Sk(b_3) \land b_1 < b_3 < b_2))\)

- **(EC2)** \((\forall b_1, d)(\exists b_2)((Sk(b_1) \land D(d) \land b_1 < d \land b_1^* \parallel d) \rightarrow (Sk(b_2) \land b_1 < b_2 \parallel d \land b_2 < 1 \land b_1 \lor b_2^* < d \land b_1^* \land b_2 \parallel d))\)

- **(EC3)** \((\exists d)(D(d) \land d < 1)\)
A finite axiomatization of $PCS^{ec}$

The list of axioms

- (EC1) $(\forall b_1, b_2)(\exists b_3)((\text{Sk}(b_1) \land \text{Sk}(b_2) \land b_1 < b_2) \rightarrow (\text{Sk}(b_3) \land b_1 < b_3 < b_2))$

- (EC2) $(\forall b_1, d)(\exists b_2)((\text{Sk}(b_1) \land D(d) \land b_1 < d \land b_1^* \parallel d) \rightarrow (\text{Sk}(b_2) \land b_1 < b_2 \parallel d \land b_2 < 1 \land b_1 \vee b_2^* < d \land b_1^* \land b_2 \parallel d))$

- (EC3) $(\exists d)(D(d) \land d < 1)$

- (EC4) $(\forall d_1, d_2)(\exists d_3)((D(d_1) \land d_1 < d_2) \rightarrow (d_1 < d_3 < d_2))$
A finite axiomatization of $PCS^{ee}$

The list of axioms

- **(EC1)** $(\forall b_1, b_2)(\exists b_3)((Sk(b_1) \& Sk(b_2) \& b_1 < b_2) \rightarrow (Sk(b_3) \& b_1 < b_3 < b_2))$

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- **(EC3)** $(\exists d)(D(d) \& d < 1)$

- **(EC4)** $(\forall d_1, d_2)(\exists d_3)((D(d_1) \& d_1 < d_2) \rightarrow (d_1 < d_3 < d_2))$

- **(EC5)** $(\forall b, d_1)(\exists d_2)((D(d_1) \& Sk(b) \& 0 < b < d_1) \rightarrow (D(d_2) \& d_2 < d_1 \& b \parallel d_2 \& d_1 \land b^* = d_2 \land b^*))$
A finite axiomatization of $PCS^{ec}$

The theorem
The theorem

At last, a finite axiomatization of $PCS^{ec}$:

(−,2012) Let $\mathcal{A}$ be a finite list of first-order sentences axiomatizing $PCS$. Then the finite set $\mathcal{A} \cup \{(AC1), \ldots, (AC4), (EC1), \ldots, (EC5)\}$ axiomatizes $PCS^{ec}$. 