Type Classification of Unification Problems over Distributive Lattices and De Morgan Algebras

Simone Bova

Vanderbilt University (Nashville TN, USA)

joint work with Leonardo Cabrer

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Outline

**Background**
- Algebraic Unification
- Distributive Lattices

**Contribution**
- Involutive Distributive Lattices
- De Morgan Classification
- Kleene Classification
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Background
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Unification Types

Let $P = (P, \leq)$ be a preorder.

A complete set for $P$ is a set $M \subseteq P$ such that:

(i) $x \parallel y$ for all $x, y \in M$ such that $x \neq y$;

(ii) for every $x \in P$ there exists $y \in M$ such that $x \leq y$.

The type of a preorder $P$ is defined by:

$$\text{type}(P) = \begin{cases} 
0, & \text{if } P \text{ has no complete set}, \\
\infty, & \text{if } P \text{ has a complete set of infinite cardinality}, \\
p, & \text{if } P \text{ has a finite complete set of cardinality } p.
\end{cases}$$
Symbolic Unification

**Problem**  \textsc{Symbequnif}(\mathcal{V})

**Instance**  A finite set \( E \subseteq \mathbb{T}_\mathcal{V}(n)^2 \).

**Solution**  \( \zeta : \{x_1, \ldots, x_n\} \rightarrow \mathbb{T}_\mathcal{V} \) such that for all \( A \in \mathcal{V} \),

\[
A \models \bigwedge_{(s,t) \in E} s(\zeta(x_1), \ldots, \zeta(x_n)) = t(\zeta(x_1), \ldots, \zeta(x_n)).
\]
**Symbolic Unification**

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*Type*  \( \text{type}_\mathcal{V}(E) = \text{type}(\mathcal{U}_\mathcal{V}(E)) \),

where preorder \( \mathcal{U}_\mathcal{V}(E) = (\mathcal{U}_\mathcal{V}(E), \leq) \) is defined by:

1. \( \mathcal{U}_\mathcal{V}(E) = \{ \zeta \mid \zeta \text{ solution to } E \} \);
2. \( \zeta_1 \leq \zeta_2 \) iff there exists \( \varsigma: \mathbb{T}_\mathcal{V} \to \mathbb{T}_\mathcal{V} \) st for all \( A \in \mathcal{V} \),

\[
A \models \bigwedge_{i \in [n]} \zeta_1(x_i) = \varsigma \circ \zeta_2(x_i).
\]
Ghilardi Algebraic Unification

**Problem** \textsc{AlgEquUnif}(\mathcal{V})

**Instance** A finitely presented algebra \( A \in \mathcal{V} \).

**Solution** A \( \sigma \)-homomorphism \( h : A \rightarrow \mathcal{P} \) such that \( \mathcal{P} \in \mathcal{V} \) is finitely presented projective.
Problem  \textsc{AlgEqUnif}(\mathcal{V})

Instance  A finitely presented algebra \( \mathbb{A} \in \mathcal{V} \).

Solution  A \( \sigma \)-homomorphism \( h: \mathbb{A} \to \mathbb{P} \) such that \( \mathbb{P} \in \mathcal{V} \) is finitely presented projective.

Type  \( \text{type}_\mathcal{V}(\mathbb{A}) = \text{type}(U_\mathcal{V}(\mathbb{A})) \),

where preorder \( U_\mathcal{V}(\mathbb{A}) = (U_\mathcal{V}(\mathbb{A}), \leq) \) is defined by:

(i)  \( U_\mathcal{V}(\mathbb{A}) = \{ h \mid h \text{ solution to } \mathbb{A} \} \);

(ii)  \( h_1 \leq h_2 \) iff there exists \( \sigma \)-hom \( f \) st \( h_1 = f \circ h_2 \).
Theorem (Ghilardi [6])
If $E \subseteq T(V)(n)^2$ finitely presents $A \in V$, then $\text{type}_V(E) = \text{type}_V(A)$. 
Ghilardi Algebraic Unification

**Theorem (Ghilardi [6])**

If \( E \subseteq T_\mathcal{V}(n)^2 \) finitely presents \( \mathbb{A} \in \mathcal{V} \), then \( \text{type}_\mathcal{V}(E) = \text{type}_\mathcal{V}(\mathbb{A}) \).

**Proof (Idea).**

Using that \( P \in \mathcal{V} \) is finitely presented projective iff
\( P \) is a retract of \( F_\mathcal{V}(n) \) for some \( n < \omega \),
prove that \( U_\mathcal{V}(E) \) and \( U_\mathcal{V}(\mathbb{A}) \) are equivalent categories.
Ghilardi Algebraic Unification

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Proof (Sketch).
For \( \zeta : \{x_1, \ldots, x_n\} \rightarrow \mathbb{T}_\mathcal{V}(m) \) solution to \( E \), define \( u_\zeta : A \rightarrow \mathbb{F}_\mathcal{V}(m) \) solution to \( A \) by \( u_\zeta([t]) = [\zeta(t)] \).

(i) For \( u : A \rightarrow P \) any solution to \( A \) with \( g : P \rightarrow \mathbb{F}_\mathcal{V}(l) \), \( f : \mathbb{F}_\mathcal{V}(l) \rightarrow P \) st \( f \circ g = \text{id}_P \), let \( \zeta : \{x_1, \ldots, x_n\} \rightarrow \mathbb{T}_\mathcal{V}(m) \) be the solution to \( E \) st
\( g(u([x_i])) = [\zeta(x_i)] \). Prove that \( u \leq u_\zeta \) and \( u_\zeta \leq u \) in \( U_\mathcal{V}(A) \).

(ii) Prove that \( \zeta_1 \leq \zeta_2 \) in \( U_\mathcal{V}(E) \) iff \( u_{\zeta_1} \leq u_{\zeta_2} \) in \( U_\mathcal{V}(A) \).
Figure: $\mathbb{L} \in \mathcal{D}\mathcal{L}$ is finitely presented by $E = \{w \lor x = y \land z\}$, then $\text{type}_{\mathcal{D}\mathcal{L}}(E) = \text{type}_{\mathcal{D}\mathcal{L}}(\mathbb{L})$. 
Ghilardi Algebraic Unification

Features:
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(i) unification is defined in terms of the categorical notions of finite presentation and projectivity, then the unification type is preserved under categorical equivalence;
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(i) unification is defined in terms of the categorical notions of finite presentation and projectivity, then the unification type is preserved under categorical equivalence;

(iii) for locally finite varieties with nice duality theorems, unification theory can be developed in the combinatorial category dual to finite algebras (in particular, the characterization of projective algebras).
Distributive Lattices | (Dual) Unification

Theorem (Birkhoff [3])

(i) Finite bounded distributive lattices and finite posets are dually equivalent (via contravariant functors $J_{DL}$ and $D_{DL}$).

Figure: $P = J_{DL}(L)$ and $L = D_{DL}(P)$. 
Theorem (Birkhoff [3], Balbes and Horn [1])

(i) Finite bounded distributive lattices and finite posets are dually equivalent (via contravariant functors $J_{DL}$ and $D_{DL}$).

(ii) A finite bounded distributive lattice $\mathbb{L}$ is projective iff $J_{DL}(\mathbb{L})$ is a finite nonempty lattice.

*Figure:* $\mathbb{P} = J_{DL}(\mathbb{L})$ and $\mathbb{L} = D_{DL}(\mathbb{P})$. $\mathbb{L}$ is not projective in $\mathcal{DL}$. 
**Problem**  \( \text{EQUINF}(\mathcal{DL}) \)

**Instance**  A finite poset \( \mathbf{P} = (P, \leq) \).

**Solution**  A \( \{\leq\}\)-homomorphism \( u : \mathbf{L} \rightarrow \mathbf{P} \),
    where \( \mathbf{L} \) is a finite nonempty lattice.
Distributive Lattices | (Dual) Unification

**Problem** \( \text{EQUNIF}(\mathcal{DL}) \)

**Instance** A finite poset \( \mathbf{P} = (P, \leq) \).

**Solution** A \( \{\leq\} \)-homomorphism \( u : L \to P \), where \( L \) is a finite nonempty lattice.

**Type** \( \text{type}_{\mathcal{DL}}(\mathbf{P}) = \text{type}(U_{\mathcal{DL}}(\mathbf{P})) \), where preorder \( U_{\mathcal{DL}}(\mathbf{P}) = (U_{\mathcal{DL}}(\mathbf{P}), \leq) \) is defined by:

\begin{enumerate}
  \item \( U_{\mathcal{DL}}(\mathbf{P}) = \{ u \mid u \text{ solution to } \mathbf{P} \} \);
  \item \( u_1 \leq u_2 \) iff there exists \( \{\leq\}\)-hom \( f\) st \( u_1 = u_2 \circ f \).
\end{enumerate}
Fact

\( P \) is a solvable instance of \( \text{UNIF}(\mathcal{D}\mathcal{L}) \) iff \( P \neq \emptyset \).

Theorem ([4])

Let \( P \) be a solvable instance of \( \text{UNIF}(\mathcal{D}\mathcal{L}) \). Then:

\[
\text{type}_{\mathcal{D}\mathcal{L}}(P) = \begin{cases} 
  p, & \text{if every interval in } P \text{ is a lattice,} \\
  0, & \text{and } P \text{ has exactly } p \text{ maximal (wrt } \subseteq \text{) intervals;} \\
\end{cases}
\]
All intervals in $\mathbf{P}$ are lattices, $\mathbf{P}$ has $p$ maximal intervals $\Rightarrow \text{type}_{\mathcal{DL}}(\mathbf{P}) = p$:

**Figure:** For all unifiers $u : \mathbf{L} \to \mathbf{P}$, there exists an inclusion map $i$ of a maximal interval $[x, y] \subseteq P$ into $\mathbf{P}$ such that $u \leq i$ in $\mathbf{U}_{\mathcal{DL}}(\mathbf{P})$. 
There exists an interval in \( P \) that is not a lattice \( \Rightarrow \) type_{\mathcal{DL}}(P) = 0:

\[
\text{Figure: For all } i < \omega, \text{ uniformly construct a unifier } u_i : G_i \rightarrow P \text{ such that, if the unifier } u : L \rightarrow P \text{ satisfies } u_i \leq u, \text{ then } |L| \geq i.
\]
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- De Morgan Classification
- Kleene Classification
De Morgan Algebras [7]

An algebra $\mathbb{A} = (A, \land, \lor, '0, 1)$ is a De Morgan algebra if:

(i) $(A, \land, \lor, 0, 1)$ is a bounded distributive lattice;
(ii) $\mathbb{A} \models x = x''$;
(iii) $\mathbb{A} \models (x \land y)' = x' \lor y'$.

Theorem (Kalman [7])
A De Morgan algebra $\mathbb{A}$ is subdirectly irreducible iff $\mathbb{A} \in \{\mathbb{B}, \mathbb{K}, \mathbb{M}\}$.

De Morgan varieties ($\mathbb{B} \subset \mathbb{K} \subset \mathbb{M}$) are locally finite.
Results

(i) Explicit characterization of injective objects in the combinatorial categories dually equivalent to finite De Morgan and Kleene algebras (key).

(ii) Complete classification of solvable instances to the (dual) De Morgan and Kleene unification problems (using the characterization),

\[ \text{type } (Q) = \begin{cases} 
1, & \text{if the "core" of } Q \text{ is injective;} \\
p < \omega, & \text{if the "core" of } Q \text{ is "almost injective;} \\
0, & \text{otherwise.} 
\end{cases} \]
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Finite De Morgan Algebras | Duality

**Objects**  Finite De Morgan algebras \( \mathbb{A} = (A, \land, \lor, ', 0, 1) \).

**Morphisms**  \{\land, \lor, ', 0, 1\}-homomorphisms.

**Objects**  Finite involutive posets (fip’s), that is, finite \{\leq, '\}\-structures \( \mathbb{P} = (P, \leq, ') \) such that,

- \( (P, \leq) \) is a partial order,
- \( \mathbb{P} \models x = x'' \), and \( \mathbb{P} \models x \leq y \) implies \( y' \leq x' \).

**Morphisms**  \{\leq, '\}\-homomorphisms.

**Theorem (Cornish and Fowler [5])**

Finite De Morgan algebras and finite involutive posets are dually equivalent (via contravariant functors \( J_\mathcal{M} \) and \( D_\mathcal{M} \)).
Finite De Morgan Algebras | Duality

Figure: $P = J_M(A)$ and $A = D_M(P)$. 
Definition ([2])
For a cardinal $\kappa$, a poset $(Q, \leq)$ is $\kappa$-complete if for all $X \subseteq Q$, if all $Y \subseteq X$ such that $|Y| < \kappa$ have an upper bound, then $X$ has a least upper bound.

Theorem ([4])
A finite De Morgan algebra $A$ is projective iff $J_M(A) = (P, \leq,')$ satisfies:

(M$_1$) $(P, \leq)$ is a nonempty lattice;
(M$_2$) for all $x \in P$ st $x \leq x'$ there exists $y \in P$ st $x \leq y = y'$;
(M$_3$) $\{x \in P \mid x \leq x'\}$ with inherited order is 3-complete.
Finite De Morgan Algebras | Projective

Figure: $P$ fails $(M_1)$, then $D_M(P)$ is not projective.
**Figure:** $P$ fails $(M_3)$, then $D_M(P)$ is not projective.
**Definition (De Morgan Unification Core)**

The De Morgan unification core of the fip \( Q \) is the fip \( Q_m = (Q_m, \leq_m, i_m) \) st:

1. \( Q_m = \{ x, x' \in Q \mid y \leq z, x, x' \text{ for some } y, z \in Q \text{ such that } z = z' \} \);
2. \( x \leq_m y \iff x \leq y \) for all \( x, y \in Q_m \);
3. \( i_m(x) = x' \) for all \( x \in Q_m \).

**Lemma**

If \( u : P \rightarrow Q \) is a unifier for \( Q \), then \( u(P) \subseteq Q_m \).
**De Morgan Algebras | Unification Type Classification**

**Problem** \( \text{EQUNIF}(\mathcal{M}) \)

**Instance** A finite involutive poset \( Q = (Q, \leq, ') \).

**Solution** A \( \{\leq, '\}\)-homomorphism \( u : P \rightarrow Q \), where \( D_M(P) \) is a finite projective De Morgan algebra.

**Fact**
\( Q \) is a solvable instance of \( \text{EQUNIF}(\mathcal{M}) \) iff \( \{x \in Q \mid x = x'\} \neq \emptyset \).

**Theorem ([4])**
Let \( Q = (Q, \leq, ') \) be a solvable instance of \( \text{UNIF}(\mathcal{M}) \). Then:

\[
\text{type}_\mathcal{M}(Q) = \begin{cases} 
p, & \text{if every interval in } Q_m \text{ satisfies } (M_1), (M_2), (M_3), \\
0, & \text{otherwise.}
\end{cases}
\]
**Figure**: $Q_m$ has poset $Q_1$ (on the left). For $i < \omega$, construct $u_i: G_i \rightarrow Q_1$ such that, if the unifier $u: L \rightarrow P$ satisfies $u_i \leq u$, then $|L| \geq i$ (on the right, $G_3$).
**De Morgan Classification** | $\mathbf{Q} \not\models (M_2)$ Gadget

*Figure:* $\mathbf{Q}_m$ has poset $\mathbf{Q}_2$ (on the left). For $i < \omega$, construct $u_i: \mathbf{G}_i \rightarrow \mathbf{Q}_2$ such that, if the unifier $u: \mathbf{L} \rightarrow \mathbf{P}$ satisfies $u_i \leq u$, then $|L| \geq i$ (on the right, $\mathbf{G}_3$).
De Morgan Classification \( \mathbb{Q} \not\models (M_3) \) Gadget

*Figure:* \( \mathbb{Q}_m \) has poset \( \mathbb{Q}_3 \) (on the left). For \( i < \omega \), construct \( u_i : G_i \rightarrow \mathbb{Q}_3 \) such that, if the unifier \( u : L \rightarrow P \) satisfies \( u_i \leq u \), then \( |L| \geq i \) (on the right, \( G_4 \)).
Finite Kleene Algebras | Duality and Projective

**Theorem (Cornish and Fowler [5])**

Finite Kleene algebras and finite involutive posets st \( x \leq x' \) or \( x' \leq x \) (kfip’s) are dually equivalent (via contravariant functors \( J_K \) and \( D_K \)).

**Theorem ([4])**

A finite Kleene algebra \( \mathbb{A} \) is projective iff \( J_K(\mathbb{A}) = (P, \leq', ') \) satisfies:

1. \( (K_1) \) \( \{x \in P \mid x \leq x'\} \) with inherited order is a nonempty meet semilattice;
2. \( (K_2) \) for all \( x, y \in P \) st \( x, y \leq y', x' \) there exists \( z \in P \) st \( x, y \leq z \leq z' \);
3. \( (M_2) \) for all \( x \in P \) st \( x \leq x' \) there exists \( y \in P \) st \( x \leq y = y' \);
4. \( (M_3) \) \( \{x \in P \mid x \leq x'\} \) with inherited order is 3-complete.
**Kleene Algebras | Unification Core**

**Definition (Kleene Unification Core)**

The *Kleene unification core* of the kfp $Q$ is the kfp $Q_k = (Q_k, \leq_k, i_k)$ st:

(i) $Q_k = \{x, x' \in Q \mid x \leq z = z' \text{ for some } z \in Q\}$;

(ii) $x \leq_k y$ iff, $x \leq y$ and either of the following three cases occurs:

(a) $x \leq x'$ and $y \leq y'$;
(b) $x' \leq x$ and $y' \leq y$;
(c) $x \leq z = z' \leq y$ for some $z \in Q$;

(iii) $i_k(x) = x'$ for all $x \in Q_k$.

**Lemma**

(i) If $u : P \rightarrow Q$ unifies $Q$, then $u(P) \subseteq Q_k$ and $u : P \rightarrow Q_k$ unifies $Q_k$.

(ii) $Q_k$ satisfies $(K_2)$ and $(M_2)$.
Kleene Algebras | Unification Type Classification

**Problem** \( \text{UNIF}(\mathcal{K}) \).

**Instance** A finite involutive poset \( Q = (Q, \leq, ') \) st \( x \leq x' \) or \( x' \leq x \).

**Solution** A homomorphism \( u: P \to Q \),
where \( D_\mathcal{K}(P) \) is a finite projective Kleene algebra.

**Fact**
\( Q \) is a solvable instance of \( \text{UNIF}(\mathcal{K}) \) iff \( \{ x \in Q \mid x = x' \} \neq \emptyset \).

**Theorem ([4])**
Let \( Q = (Q, \leq, ') \) be a solvable instance of \( \text{UNIF}(\mathcal{K}) \). Then:

\[
\text{type}_\mathcal{K}(Q) = \begin{cases} 
p, & \text{if every interval in } Q_k \text{ satisfies } (K_1) \text{ and } (M_3), \\
& \text{and } Q_k \text{ has exactly } p \text{ maximal intervals}; \\
0, & \text{otherwise}. 
\end{cases}
\]
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Thank you for your attention!