Math 215 - Integrals

Practice Integrals - Solutions

1.

$$\begin{aligned}\int_0^3 (5x^2 - 4x + 3)dx &= 5(\frac{x^3}{3}) - 4(\frac{x^2}{2}) + 3x\Big|_0^3 \\ &= \frac{5x^3}{3} - 2x^2 + 3x\Big|_0^3 \\ &= 5 \cdot 9 - 2 \cdot 9 + 9 = 45 - 9 = 36\end{aligned}$$

2.

$$\begin{aligned}\int_1^3 (x + \frac{1}{x})^2 dx &= \int_1^3 (x^2 + 2 + \frac{1}{x^2})dx \\ &= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1}\Big|_1^3 \\ &= \frac{x^3}{3} + 2x - \frac{1}{x}\Big|_1^3 \\ &= 9 + 6 - \frac{1}{3} - (\frac{1}{3} + 2 - 1) \\ &= 15 - \frac{1}{3} - \frac{1}{3} - 1 = \frac{40}{3}\end{aligned}$$

3.

$$\begin{aligned}\int \frac{x^4 - 1}{x^2 + 1} dx &= \int \frac{(x^2 + 1)(x^2 - 1)}{x^2 + 1} dx \\ &= \int (x^2 - 1)dx \\ &= \frac{x^3}{3} - x + C\end{aligned}$$

4.

$$\begin{aligned}\int \frac{t^6 - t^4}{t^4} dt &= \int (t^2 - 1)dt \\ &= t^3 - t + C\end{aligned}$$

5.

$$\begin{aligned}\int (x^3 - 1)^2 dx &= \int ((x^3)^2 - 2x^3 + 1)dx \\ &= \int (x^6 - 2x^3 + 1)dx \\ &= \frac{x^7}{7} - 2\frac{x^4}{4} + x + C = \frac{x^7}{7} - \frac{x^4}{2} + x + C\end{aligned}$$

6.

$$\begin{aligned}\int_2^4 (x-1)(3x+2)dx &= \int_2^4 (3x^2-x-2)dx \\ &= x^3 - \frac{x^2}{2} - 2x\Big|_2^4 \\ &= 64-8-8-(8-2-4) = 46\end{aligned}$$

7.

$$\begin{aligned}\int_0^1 (\sqrt[4]{x^5}+\sqrt[5]{x^4})dx &= \int_0^1 (x^{\frac{5}{4}}+x^{\frac{4}{5}})dx \\ &= \frac{4x^{\frac{9}{4}}}{9} + \frac{5x^{\frac{9}{5}}}{9}\Big|_0^1 \\ &= \frac{4}{9}+\frac{5}{9}-0 = 1\end{aligned}$$

8.

$$\begin{aligned}\int_0^1 x^{\frac{1}{2}}x^3 dx &= \int_0^1 x^{\frac{7}{2}}dx \\ &= \frac{2x^{\frac{9}{2}}}{9}\Big|_0^1 = \frac{9}{2}\end{aligned}$$

9. $\int_0^{(\frac{\pi}{2})^2} \frac{\sin\sqrt{x}}{\sqrt{x}}dx$

Let $u=\sqrt{x}$, then $du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$. Substitute and get:

$$\begin{aligned}\int_0^{(\frac{\pi}{2})^2} \frac{\sin\sqrt{x}}{\sqrt{x}}dx &= 2\int_{x=0}^{(\frac{\pi}{2})^2} \sin u du \\ &= -2\cos u\Big|_{x=0}^{x=(\frac{\pi}{2})^2} \\ &= -2\cos\sqrt{x}\Big|_0^{(\frac{\pi}{2})^2} \\ &= -2\cdot 0 - (-2(1)) = 2\end{aligned}$$

10. $\int_0^1 \frac{4}{(1+2x)^3}dx$

Let $u = 1+2x$, then $du = 2dx$ and $dx = \frac{1}{2}du$. Substitute and get:

$$\begin{aligned}\int_0^1 \frac{4}{(1+2x)^3}dx &= 4\cdot\frac{1}{2}\int_{x=0}^{x=1} \frac{1}{u^3}du \\ &= 2\int_{x=0}^{x=1} u^{-3}du \\ &= 2\frac{u^{-2}}{-2}\Big|_{x=0}^{x=1} = -\frac{1}{u^2}\Big|_{x=0}^{x=1} \\ &= \frac{-1}{(1+2x)^2}\Big|_0^1 = \frac{-1}{3^2} - (-\frac{1}{1}) = \frac{1}{9}+1 = \frac{8}{9}\end{aligned}$$

11. $\int 2x(x^2+3)^4 dx$

Let $u = x^2+3$, then $du = 2xdx$. Substitute and get:

$$\begin{aligned} \int 2x(x^2+3)^4 dx &= \int u^4 du \\ &= \frac{u^5}{5} + C = \frac{(x^2+3)^5}{5} + C \end{aligned}$$

12. $\int x^3(1-x^4)^5 dx$

Let $u = 1-x^4$, then $du = -4x^3$ and $x^3dx = -\frac{1}{4}du$. Substitute:

$$\begin{aligned} \int x^3(1-x^4)^5 dx &= -\frac{1}{4}\int u^5 du \\ &= -\frac{1}{4}\cdot\frac{u^6}{6} + C = -\frac{(1-x^4)^6}{24} + C \end{aligned}$$

13. $\int_1^2 \sqrt{x-1}dx$

Let $u = x-1$, then $du = dx$.

$$\begin{aligned} \int_1^2 \sqrt{x-1}dx &= \int_{x=1}^{x=2} u^{\frac{1}{2}} du \\ &= \frac{2u^{\frac{3}{2}}}{3}\Big|_{x=1}^{x=2} \\ &= \frac{2(x-1)^{\frac{3}{2}}}{3}\Big|_1^2 = \frac{2}{3} - 0 = \frac{2}{3} \end{aligned}$$

14. $\int x(x^2+1)^{\frac{3}{2}} dx$

Let $u = x^2+1$, then $du = 2xdx$ and $\frac{1}{2}du = xdx$. Substitute:

$$\begin{aligned} \int x(x^2+1)^{\frac{3}{2}} dx &= \frac{1}{2}\int u^{\frac{3}{2}} du \\ &= \frac{1}{2}\cdot\frac{2u^{\frac{5}{2}}}{5} + C \\ &= \frac{1}{5}(x^2+1)^{\frac{5}{2}} + C \end{aligned}$$

15. $\int \frac{1}{(1-3t)^4} dt$

Let $u = 1-3t$, then $du = -3dt$ and $-\frac{1}{3}du = dt$. Substitute:

$$\begin{aligned}
\int \frac{1}{(1-3t)^4}dt &= -\frac{1}{3}\int \frac{1}{u^4}du \\
&= -\frac{1}{3}\int u^{-4}du \\
&= -\frac{1}{3}\cdot\frac{u^{-3}}{-3}+C \\
&= \frac{1}{9}(1-3t)^{-3}+C
\end{aligned}$$

16. $\int \sqrt{3-5y}dy$

Let $u = 3-5y$, then $du = -5dy$ and $-\frac{1}{5}du = dy$. Substitute:

$$\begin{aligned}
\int \sqrt{3-5y}dy &= -\frac{1}{5}\int \sqrt{u}du \\
&= -\frac{1}{5}\int u^{\frac{1}{2}}du \\
&= -\frac{1}{5}\cdot\frac{2}{3}u^{\frac{3}{2}}+C \\
&= -\frac{2}{15}(3-5y)^{\frac{3}{2}}+C
\end{aligned}$$

17. $\int (\ln x)^2 dx$ Hint: use integration by parts two times.

Set up IBP (Integration By Parts):

$$\begin{aligned}
u = (\ln x)^2 &\quad dv = dx \\
du = 2(\ln x)\frac{1}{x}dx &\quad v = x
\end{aligned}$$

Now,

$$\begin{aligned}
\int (\ln x)^2 dx &= x(\ln x)^2 - 2\int (\ln x)\cdot\frac{1}{x}\cdot x dx \\
&= x(\ln x)^2 - 2\int \ln x dx
\end{aligned}$$

Apply IBP a second time to find $\int \ln x dx$.

$$\begin{aligned}
u = \ln x &\quad dv = dx \\
du = \frac{1}{x}dx &\quad v = x
\end{aligned}$$

Now,

$$\begin{aligned} \int \ln x dx &= x \ln x - \int \frac{1}{x} \cdot x dx \\ &= x \ln x - \int dx = x \ln x - x \end{aligned}$$

Finally we have,

$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x dx = x(\ln x)^2 - 2[x \ln x - x] + C$

18. $\int t^3 e^t dt$ Hint: use integration by parts three times.

Set up IBP:

$$\begin{aligned} u = t^3 \quad & dv = e^t dt \\ du = 3t^2 dt \quad & v = e^t \end{aligned}$$

Now,

$$\int t^3 e^t dt = t^3 e^t - 3\int t^2 e^t dt$$

Apply IBP a second time to find $\int t^2 e^t dt$:

$$\begin{aligned} u = t^2 \quad & dv = e^t \\ du = 2t dt \quad & v = e^t \end{aligned}$$

We get,

$$\int t^2 e^t dt = t^2 e^t - 2\int t e^t dt$$

Apply IBP a third time to find $\int t e^t dt$:

$$\begin{aligned} u = t \quad & dv = dt \\ du = dt \quad & v = t \end{aligned}$$

and,

$$\begin{aligned} \int t e^t dt &= t e^t - \int e^t dt \\ &= t e^t - e^t \end{aligned}$$

Finally, we have the integral,

$$\begin{aligned}\int t^3e^tdt &= t^3e^t - 3(\int t^2e^tdt) \\ &= t^3e^t - 3(t^2e^t - 2\int te^tdt) \\ &= t^3e^t - 3(t^2e^t - 2(te^t - \int e^tdt)) + C \\ &= t^3e^t - 3(t^2e^t - 2(te^t - e^t)) + C\end{aligned}$$

19. $\int xe^{2x}dx$

Set up IBP:

$$\begin{array}{rr} u = x & dv = e^{2x}dx \\ du = dx & v = \frac{1}{2}e^{2x} \end{array}$$

Thus,

$$\begin{aligned}\int xe^{2x}dx &= \frac{1}{2}\cdot x \cdot e^{2x} - \frac{1}{2}\int e^{2x}dx \\ &= \frac{1}{2}\cdot x\cdot e^{2x} - \frac{1}{2}\cdot\frac{1}{2}\cdot e^{2x} + C = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\end{aligned}$$

20. $\int x\cos xdx$

Set up IBP:

$$\begin{array}{rr} u = x & dv = \cos xdx \\ du = dx & v = \sin x \end{array}$$

Thus,

$$\begin{aligned}\int x\cos xdx &= x\sin x - \int \sin xdx \\ &= x\sin x - (-\cos x) + C = x\sin x + \cos x + C\end{aligned}$$

21. $\int_0^1 xe^{-x}dx$

Set up IBP:

$$\begin{array}{rr} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{array}$$

Thus,

$$\begin{aligned}
\int_0^1 xe^{-x}dx &= -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx \\
&= -xe^{-x} + (-e^{-x})\Big|_0^1 \\
&= (-1)e^{-1} - e^{-1} - (0 - e^0) = -\frac{2}{e} + 1
\end{aligned}$$

22. $\int \frac{\ln x}{x^2} dx$

Set up IBP:

$$\begin{aligned}
u = \ln x \quad & dv = x^{-2}dx \\
dv = \frac{1}{x}dx \quad & v = -x^{-1}
\end{aligned}$$

Thus,

$$\begin{aligned}
\int \frac{\ln x}{x^2} dx &= -\frac{1}{x}\ln x + \int \frac{1}{x^2}dx \\
&= -\frac{1}{x}\ln x - x^{-1} + C
\end{aligned}$$

23. $\int x \ln x dx$

Set up IBP:

$$\begin{aligned}
u = \ln x \quad & dv = xdx \\
du = \frac{1}{x}dx \quad & v = \tfrac{x^2}{2}
\end{aligned}$$

Thus,

$$\begin{aligned}
\int x \ln x dx &= \frac{x^2}{2}\ln x - \frac{1}{2}\int \frac{x^2}{2}dx \\
&= \frac{x^2}{2}\ln x - \frac{1}{2}\int xdx \\
&= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C \\
&= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C
\end{aligned}$$

24. $\int x^2 \cos x dx$ Hint: use integration by parts two times.

Set up IBP:

$$\begin{aligned} u = x^2 \quad & dv = \cos x dx \\ du = 2x dx \quad & v = \sin x \end{aligned}$$

Thus,

$$\int x^2 \cos x dx = x^2 \sin x - 2\int x \sin x dx$$

Apply IBP a second time to find $\int x \sin x dx$,

$$\begin{aligned} u = x \quad & dv = \sin x dx \\ du = dx \quad & v = -\cos x \end{aligned}$$

So,

$$\begin{aligned} \int x \sin x dx &= -x\cos x + \int \cos x dx \\ &= -x\cos x + \sin x \end{aligned}$$

Therefore,

$$\begin{aligned} \int x^2 \cos x dx &= x^2 \sin x - 2(\int x \sin x dx) \\ &= x^2 \sin x - 2(-x\cos x + \sin x) + C \end{aligned}$$

25. $\int \sqrt{x} \ln x dx$

Set up IBP:

$$\begin{aligned} u = \ln x \quad & dv = \sqrt{x} dx \\ du = \frac{1}{x} dx \quad & v = \tfrac{2}{3} x^{\frac{3}{2}} \end{aligned}$$

Thus,

$$\begin{aligned} \int \sqrt{x} \ln x dx &= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \int \frac{x^{\frac{3}{2}}}{x} dx \\ &= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \int x^{\frac{1}{2}} dx \\ &= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3}(\frac{2}{3} x^{\frac{3}{2}}) + C = \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{4}{9} x^{\frac{3}{2}} + C \end{aligned}$$

26. Problem 17 section 6.3 $\int_1^4 e^{\sqrt{x}} dx$

Since the bounds of integration are from 1 to 4 we can multiply the function on the inside of the integral by $\frac{\sqrt{x}}{\sqrt{x}}$. Thus,

$\int_1^4 e^{\sqrt{x}} dx = \int_1^4 e^{\sqrt{x}} \frac{\sqrt{x}}{\sqrt{x}} dx.$

We now use substitution: Let $u = \sqrt{x}$, then $du = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} dx$ and $2du = \frac{1}{\sqrt{x}} dx$.

Thus,

$$\int_1^4 e^{\sqrt{x}} \frac{\sqrt{x}}{\sqrt{x}} dx = \int_{x=1}^{x=4} ue^u du$$

We now apply IBP to find $\int_{x=1}^{x=4} ue^u du$:

$$\begin{aligned} w &= u & dv &= e^u du \\ dw &= du & v &= e^u \end{aligned}$$

Thus,

$$\begin{aligned} \int_{x=1}^{x=4} ue^u du &= ue^u - \int_{x=1}^{x=4} e^u du \\ &= ue^u - e^u \Big|_{x=1}^{x=4} \end{aligned}$$

Substituting $u = \sqrt{x}$ back into the equations yields:

$$\begin{aligned} \int_1^4 e^{\sqrt{x}} \frac{\sqrt{x}}{\sqrt{x}} dx &= \int_{x=1}^{x=4} ue^u du \\ &= ue^u - e^u \Big|_{x=1}^{x=4} \\ &= \sqrt{x} e^{\sqrt{x}} - e^{\sqrt{x}} \Big|_1^4 \\ &= 2e^2 - e^2 - (e^1 - e^1) = e^2 - 0 = e^2 \end{aligned}$$