Section 7.5

# 16

Solve the differential equation

\[(\sec x) \frac{dy}{dx} = e^{y + \sin x}\]

**Solution:** By separating the variables,

\[e^{-y} dy = \cos(x) e^{\sin(x)} dx\]

The left side is straightforward to integrate. The right side can be done using \(u\)-substitution with \(u = \sin(x)\). This gives:

\[-e^{-y} = e^{\sin(x)} + C\]
\[e^{-y} = -e^{\sin(x)} - C\]
\[-y = \ln \left( -e^{\sin(x)} - C \right)\]
\[y = -\frac{\ln(-e^{\sin(x)})}{\ln(C)}\]

□

# 20

The earth’s atmospheric pressure \(p\) is often modeled by assuming that the rate \(dp/dh\) at which \(p\) changes with the altitude \(h\) above sea level is proportional to \(p\). Suppose that the pressure at sea level is 1013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.

(a) Solve the initial value problem:

Differential equation: \(dh/dp = kp\), \((k\) a constant\)
Initial condition: \(p = p_0\) when \(h = 0\)

to express \(p\) in terms of \(h\). Determine the values of \(p_0\) and \(k\) from the given altitude-pressure data.

(b) What is the atmospheric pressure at \(h = 50\) km?

(c) At what altitude does the pressure equal 900 millibars?

**Solution:**

(a) By separating variables or by general knowledge of exponential growth,

\[p(h) = p_0 e^{kh}\]

and \(p(0) = 1013 = p_0\) and \(p(20) = 90\). Solving, we have

\[90 = 1013 e^{20k}\]

\[\ln \left( \frac{90}{1013} \right) = 20k\]
\[\frac{\ln(90/1013)}{20} = k\]
(b) \( p(50) = 2.38 \) millibars.

c) We must solve for \( h \):

\[
900 = 1013e^{kh} \\
\ln \left( \frac{900}{1013} \right) = kh \\
h = 0.97715 \text{ km}
\]

Section 7.6

# 22

Use l’Hôpital’s rule to find the limit.

\[
\lim_{x \to \pi/2} \frac{\ln(\csc x)}{(x - (\pi/2))^2}
\]

**Solution:** By l’Hôpital’s rule, the limit above is equal to

\[
\lim_{x \to \pi/2} \frac{-\csc x \cot x}{2(x - (\pi/2))} = \lim_{x \to \pi/2} \frac{-\cot x}{2(x - (\pi/2))}
\]

Differentiating once more,

\[
\lim_{x \to \pi/2} \frac{\csc^2 x}{2} = \frac{1}{2}
\]

# 32

Use l’Hôpital’s rule to find the limit.

\[
\lim_{x \to \infty} \frac{\log_2 x}{\log_3(x + 3)}
\]

**Solution:** I prefer logarithms in terms of \( e \), so first I rewrite the problem in base \( e \):

\[
\lim_{x \to \infty} \frac{\ln x}{\ln(x + 3)}
\]

which is

\[
\lim_{x \to \infty} \frac{\ln 3}{\ln 2} \cdot \frac{\ln x}{\ln(x + 3)}
\]

Now by l’Hôpital’s rule, the limit above is equal to

\[
\lim_{x \to \infty} \frac{\ln 3}{\ln 2} \cdot \frac{\frac{1}{x}}{\frac{1}{x+3}} = \lim_{x \to \infty} \frac{\ln 3}{\ln 2} \cdot \frac{x+3}{x} = \frac{\ln 3}{\ln 2}
\]
Use l’Hôpital’s rule to find the limit.

\[
\lim_{x \to 0^+} \left( \frac{3x + 1}{x} - \frac{1}{\sin x} \right)
\]

**Solution:** First, combine into one fraction:

\[
\lim_{x \to 0^+} \left( \frac{(3x + 1) \sin(x) - x}{x \sin(x)} \right) = \lim_{x \to 0^+} \frac{3x \sin(x) + \sin(x) - x}{x \sin(x)}
\]

By l’Hôpital’s rule, we differentiate:

\[
\lim_{x \to 0^+} \frac{3x \cos(x) + 3 \sin(x) + \cos(x) - 1}{x \cos(x) + \sin(x)}
\]

We still get an indeterminant form, and must differentiate again:

\[
\lim_{x \to 0^+} \frac{-3x \sin(x) + 3 \cos(x) + 3 \cos(x) - \sin(x)}{-x \sin(x) + \cos(x) + \cos(x)} = \lim_{x \to 0^+} \frac{-3x \sin(x) + 6 \cos(x) - \sin(x)}{-x \sin(x) + 2 \cos(x)} = \frac{6}{2} = 3
\]

# 58

L’Hôpital’s Rule does not help with this limit. Try it – you just keep on cycling. Find the limit some other way.

\[
\lim_{x \to 0} \frac{\sqrt{x}}{\sqrt{\sin x}}
\]

**Solution:** If we try l’Hôpital’s rule we get:

\[
\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{\sin(x)}} = \lim_{x \to 0^+} \frac{\frac{1}{2} x^{-1/2}}{(\sin(x))^{-1/2} \cos(x)}
\]

\[
= \lim_{x \to 0^+} \frac{\sqrt{\sin(x)}}{\sqrt{x} \cos(x)}
\]

\[
= \lim_{x \to 0^+} \frac{\sqrt{\sin(x)}}{\sqrt{x}}
\]

since \(\lim_{x \to 0} \cos(x) = 1\). Notice the last limit is just the reciprocal of the original; so, we are getting nowhere. Instead use properties of limits to rewrite and solve:

\[
\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{\sin(x)}} = \lim_{x \to 0^+} \sqrt{\frac{x}{\sin x}}
\]

\[
= \sqrt{\lim_{x \to 0^+} \frac{x}{\sin x}}
\]

\[
= \sqrt{1}
\]

\[
= 1
\]

\(\square\)