Section 9.3

# 6

Does the series converge or diverge? Give reasons for your answer.

$$\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$$

Solution: CONVERGES. Rewritten, the sum $S$ equals:

$$S = -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Now the terms $\{a_n = \frac{1}{n^{3/2}}\}$ are a sequence of positive terms, where $a_n = f(n) = n^{-3/2}$. The function $f(x)$ is a continuous, positive, decreasing function of $x$ for all $x \geq 1$. By the Integral Test (Theorem 9, page 517), the integral

$$\int_{1}^{\infty} x^{-3/2} dx = -2x^{-1/2}\bigg|_{1}^{\infty} = 2 - 2 \lim_{x \to \infty} x^{-1/2} = 2$$

converges, and so the series must also converge.

# 10

Does the series converge or diverge? Give reasons for your answer.

$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$$

Solution: DIVERGES. Let $S$ be the sum given above, if it exists, and define

$$T = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

Then clearly

$$T < S$$

so if $S$ exists, $T$ must also exist. By direct comparison, if $T$ diverges then $S$ must also diverge.

The terms $\{a_n = \frac{\ln n}{\sqrt{n}}\}$ are a sequence of positive terms, where $a_n = f(n) = n^{-1/2}$. The function $f(x)$ is a continuous, positive, decreasing function of $x$ for all $x \geq 2$. By the Integral Test, the integral

$$\int_{2}^{\infty} x^{-1/2} dx = 2\sqrt{x}\bigg|_{2}^{\infty} = -2\sqrt{2} + 2 \lim_{x \to \infty} \sqrt{x}$$

diverges, and so the series $T$ must also diverge, and by direct comparison, $S$ must also diverge.
# 20

Does the series converge or diverge? Give reasons for your answer.

\[ \sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n} \]

Solution: **CONVERGES.** By calculation, \( \ln(3) \approx 1.0986 > 1 \). Rewritten

\[ \sum_{n=1}^{\infty} \left( \frac{1}{\ln 3} \right)^n \]

and we have a geometric series with \( a = 1 \) and \( r = \frac{1}{\ln 3} \). Since \( |r| < 1 \) we have

\[ \sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n} = \frac{1 - \frac{1}{\ln 3}}{1 - \frac{1}{\ln 3}} = \frac{\ln 3}{\ln 3 - 1} \approx 11.14072 \]

# 22

Does the series converge or diverge? Give reasons for your answer.

\[ \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln^2 n)} \]

Solution: **CONVERGES.** The terms \( \{a_n = \frac{1}{n(1 + \ln^2 n)}\} \) are a sequence of positive terms, where \( a_n = f(n) = \frac{1}{n(1 + \ln^2 n)} \). The function \( f(x) \) is a continuous, positive, decreasing function of \( x \) for all \( x \geq 1 \). By the Integral Test (Theorem 9, page 517), the integral

\[ \int_1^{\infty} \frac{1}{x(1 + \ln x)^2} \, dx = \int_1^{\infty} \frac{1}{1 + u^2} \, du \]

\[ = \left[ \arctan(u) \right]_1^{\infty} = \arctan(\ln(x)) \]

\[ = \lim_{x \to \infty} \arctan(\ln(x)) = \lim_{x \to \infty} \arctan(x) = \pi/2 \]

converges, and so the series must also converge.

# 32

For what values of \( a \), if any, does the series converge?

\[ \sum_{n=3}^{\infty} \left( \frac{1}{n-1} - \frac{2a}{n+1} \right) \]

Solution:

It is convenient to re-index the series. Let \( m = n + 1 \), so the series is rewritten:

\[ \sum_{m=4}^{\infty} \frac{1}{m-2} - \frac{2a}{m} = \sum_{m=4}^{\infty} \frac{m - 2a(m-2)}{m(m-2)} \]
Splitting the series into the sum of two series:

\[
\sum_{m=4}^\infty \frac{m - 2a m + 4a}{m(m - 2)} = \sum_{m=4}^\infty \frac{(1 - 2a)m}{m(m - 2)} + \sum_{m=4}^\infty \frac{4a}{m(m - 2)}
\]

\[
= \sum_{m=4}^\infty \frac{1 - 2a}{m - 2} + \sum_{m=4}^\infty \frac{4a}{m(m - 2)}
\]

\[
= (1 - 2a) \sum_{m=4}^\infty \frac{1}{m - 2} + 4a \sum_{m=4}^\infty \frac{1}{m(m - 2)}
\]

First let’s consider the righthand series. By the Integral Test we can show that the series converges:

\[
\int_4^\infty \frac{1}{x(x - 2)} \, dx = \frac{1}{2} \int_4^\infty \frac{1}{x - 2} - \frac{1}{x} \, dx
\]

\[
= \frac{1}{2} \ln \left( \frac{x - 2}{x} \right) \bigg|_4^\infty
\]

\[
= -\frac{1}{2} \ln \left( \frac{2}{4} \right) + \frac{1}{2} \lim_{x \to \infty} \ln \left( \frac{x - 2}{x} \right)
\]

\[
= \ln \left( \sqrt{2} \right) + 0
\]

So, let \( c \) represent the value that the righthand series converges to. Then the original sum is equal to

\[
c + (1 - 2a) \sum_{m=4}^\infty \frac{1}{m - 2}
\]

So, the entire series will converge only if the series above converges. Since this is actually the harmonic series (again, by re-indexing), and since we know the harmonic series diverges, we must have that

\[
(1 - 2a) = 0
\]

and therefore \( a = \frac{1}{2} \) is the only value of \( a \) for which the original series converges.

Section 9.4

# 4

Does the series converge or diverge? Given reasons for your answer.

\[
\sum_{n=1}^\infty \frac{1 + \cos n}{n^2}
\]

Solution: CONVERGES. Since \( \cos n \leq 1 \) for all \( n \), we have

\[
\sum_{n=1}^\infty \frac{1 + \cos n}{n^2} \leq \sum_{n=1}^\infty \frac{2}{n^2} = 2 \sum_{n=1}^\infty \frac{1}{n^2}
\]

Since the righthand series is a \( p \)-series, with \( p = 2 \), we know it converges, and therefore the original series must also converge.

# 12

Does the series converge or diverge? Given reasons for your answer.

\[
\sum_{n=1}^\infty \frac{(\ln n)^3}{n^3}
\]
Solution: CONVERGES. Let \( a_n = \frac{(\ln n)^3}{n^3} \) and \( b_n = \frac{1}{n^2} \). Then \( \sum b_n \) is a \( p \)-series, with \( p = 2 > 1 \), so \( \sum b_n \) converges.

Then, we calculate:

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(\ln n)^3}{n} = \lim_{n \to \infty} \frac{3(\ln n)^2}{n} = \lim_{n \to \infty} \frac{6(\ln n)}{n} = \lim_{n \to \infty} \frac{6}{n} = 0
\]

By Part 2 of the Limit Comparison Test, we know that \( \sum a_n \) must converge.

\# 24

Does the series converge or diverge? Given reasons for your answer.

\[
\sum_{n=1}^{\infty} \frac{3^{n-1} + 1}{3^n}
\]

Solution: DIVERGES. In order for the series to converge, the limit of the sequence \( a_n = \frac{3^{n-1} + 1}{3^n} \) must be zero as \( n \to \infty \):

\[
\lim_{n \to \infty} \frac{3^{n-1} + 1}{3^n} = \frac{1}{3} + \lim_{n \to \infty} \frac{1}{3^n} = \frac{1}{3} \neq 0
\]

Since the sequence does not approach zero, we know the series diverges.

\# 36

Does the series converge or diverge? Given reasons for your answer.

\[
\sum_{n=1}^{\infty} \frac{1}{1 + 2^2 + 3^2 + \cdots + n^2}
\]

Solution: CONVERGES. The expression in the denominator is well-known and can be rewritten:

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)(2n+1)} = \sum_{n=1}^{\infty} \frac{6}{n(n + 1)(2n + 1)} \leq \sum_{n=1}^{\infty} \frac{6}{n^3}
\]

The rightmost series is a \( p \)-series with \( p = 3 > 1 \), so it converges, and by comparison, the original series must also converge.