MATH 413 HW 14

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1. Suppose p and q are distinct primes. Show there is no simple group of order p^2q.

Solution:

2. Let H be a normal subgroup of order p^k in a finite group G. Show that H is contained in every Sylow-p subgroup. Hint: use the second Sylow theorem and the fact (that was proved in class) that every subgroup of order p^k is contained in some Sylow-p subgroup.

Solution:

3. Let P be a Sylow-p subgroup of a finite group G. Prove that $N(N(P)) = N(P)$.

Solution:

4. Suppose $p < q < r$ are primes and $|G| = pqr$. Show that G is not simple. More is true: the Sylow-r subgroup must be normal. This is pretty hard so I will give you extra credit bonus points if you can prove this.

Solution:

5. Suppose p and q are distinct primes. Show there is no simple group of order p^2q^2.

Solution:

6. The groups of order less that 110 to which none of the above nor the p^3q result I proved in class shows that a group of that order cannot be simple are: $72 = 2^3 \cdot 3^2$, $80 = 2^4 \cdot 5$, $84 = 2^2 \cdot 3 \cdot 7$, $90 = 2 \cdot 3^2 \cdot 5$, $96 = 2^5 \cdot 3$, and $108 = 2^2 \cdot 3^3$. Show that there is no simple group of any of these orders.

Solution:

7. Let G be a group of order 60 which has a normal Sylow 3-subgroup. Prove that G also has a normal Sylow 5-group. This is #16 from section 9.3 It is the one I didn't see an easy, "B" level, solution to. If you can find an easy solution I would love to see it. But if you don't you can just skip it.

Here is an outline of my harder solution: let N be the normal subgroup of order 3 and H a subgroup of order 5. Since N is normal, $T := NH$ is a subgroup of order 15. Its index is 4 so by theorem in the Actions handout, there is a homomorphism $\varphi : G \to S_4$ with $\varphi(G)$ being a subgroup of S_4. Let K be the kernel of φ. Since any element of G not in T then $\varphi(g) = gT \neq T$, g is not in K. So $K \subseteq T$. So $|K| \leq 15$. Since S_4 has no elements of order 5, $H \subseteq K$. It follow (actually from Sylow) that $H \trianglelefteq K$. But normal Sylow subgroups are characteristic (invariant under all automorphisms). So

$$H \text{ char } K \trianglelefteq G$$

So $H \trianglelefteq G$.