MATH 412 MIDTERM MAKEUP

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1. Which of the polynomials below is irreducible in $\mathbb{Q}[x]$. If it is irreducible give a reason; if it is reducible give a factorization.

a. $x^5 - 4x + 22$

Solution:

b. $10 - 15x + 25x^2 - 7x^4$

Solution:

c. $x^3 - 24x - 5$

Solution:

2. Let $k \in \mathbb{Z}$ and let $f(x) = x^9 + 12x^5 - 21x + k$. Show that there are infinitely many integers k such that $f(x)$ is irreducible $\mathbb{Q}[x]$.

Solution:

3. Let I be an ideal of a ring R. The *left annihilator* of I is the set J, where

$$J = \{s \in R : sa = 0 \text{ for every } a \in I\}$$

Show that J is an ideal of R.

Solution:

4. Let R be a ring in which $x^2 = x$ holds for every $x \in R$.

a. Show that $2a = 0$ for all $a \in R$. (Of course $2a$ means $a+a$.)
Hint: $a + a = (a + a)^2$.

Solution:

b. Show R is commutative. **Hint:** $a + b = (a + b)^2$.

Solution:

5. Let H be a subgroup of a group G. Define a relation $a \sim b$ on G by

$$a \sim b \text{ if } a = bh \text{ for some } h \in H$$

a. Show that $\sim$ is an equivalence relation on G.

Solution:

b. Let $a \in G$. Show that the block (block means equivalence class) of this equivalence relation containing a is aH. (Recall that aH is defined as $\{ah : h \in H\}$.)

Solution:

c. Show that $|H| = |aH|$ by showing that map $f : H \to aH$ defined by $f(h) = ah$ is one-to-one and onto.

Solution:

d. Prove that if G is a finite group and H is a subgroup, then $|H|$ divides $|G|$. (This is known as Lagrange's Theorem).

Solution: