1. Let $G$ be a group and let $Z(G)$ be its center. Show $Z(G)$ is a normal subgroup of $G$.

2. The class equation. Let $G$ be a finite group. Define a relation $\sim$ on $G$ by $x \sim y$ if there exists a $g \in G$ such that $y = gxg^{-1}$.
   
   a. Show that this is an equivalence relation on $G$. Some books denote the equivalence class containing $x$ by $\text{cl}(x)$.
   
   b. Show that $|\text{cl}(x)| = 1$ if and only if $x \in Z(G)$.
   
   c. Let $x_1, \ldots, x_k$ be representatives of the classes which have more than 1 element. (This means that if $|\text{cl}(x)| > 1$ then there is exactly one $i$ such that $\text{cl}(x) = \text{cl}(x_i)$.) Show
   $$|G| = |Z(G)| + |\text{cl}(x_1)| + \cdots + |\text{cl}(x_k)|$$
   
   d. Let $C(x) = \{g \in G : gx = xg\}$. $C(x)$ is called the centralizer of $x$. Show that $C(x)$ is a subgroup of $G$ and that $C(x) = G$ if and only if $x \in Z(G)$.
   
   e. Prove that $|\text{cl}(x)| = [G : C(x)] = |G|/|C(x)|$. To do this you need to define a map from $\text{cl}(x)$ to the cosets of $C(x)$. Here’s how. If $y \in \text{cl}(x)$ then it is conjugate to $x$, so $y = gxg^{-1}$ for some $g$. The map is $y \mapsto gC(x)$. You need to show that this map is well defined, one-to-one, and onto. This is done in the book but try to prove it without looking at the book’s proof unless you really get stuck.
   
   f. Combine the stuff above to get the class equation:
   $$|G| = |Z(G)| + [G : C(x_1)] + \cdots + [G : C(x_k)]$$
   where $x_1, \ldots, x_k$ are as before.
   
   g. Suppose $G$ is a group of order $p^k$, where $p$ is a prime and $k > 0$. Show that $G$ has a nontrivial center. (This means $|Z(G)| > 1$.)

3. Let $H$ be a subgroup of $G$ of index 2. Show that $H \triangleleft G$.

4. Show that if $G/Z(G)$ is cyclic then $G$ is abelian. This was on the last homework; make sure you know how to do it.
5. Let $G$ be a finite group. Show that a nonempty subset $H$ of $G$ is a subgroup if it is closed under multiplication. Give an example of an infinite group which has a nonempty subset which is closed under multiplication but is not a subgroup.

6. Let $\varphi : G \to H$ be a map from a group $G$ into a group $H$. Show that if $\varphi(ab) = \varphi(a)\varphi(b)$, for all $a, b \in G$, then $\varphi$ is a homomorphism.

7. Let $H$ and $N$ be subgroups of a group $G$ and assume $N \triangleleft G$. Show that $HN$ is a subgroup of $G$.

8. Show that there is is no simple group of order 160, 200, 231, 255, 48, 36, 312, 251, or 96. You are allowed to use that if $G$ is a simple group and $H \leq G$ and $n = [G : H]$, then $|G|$ divides $n!$. 