

MATH 412 HW 3: September 23, 2015

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1. Let $d \in \mathbb{Z}$ be square-free. This means there is no element $a > 1$ in \mathbb{Z} such that $a^2 \mid d$. So d is square-free if and only if it is a product of distinct primes. Consider the ring

$$R = \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}.$$

Define the *norm* of an element by

$$N(a + b\sqrt{d}) = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - b^2d.$$

- a. Show that if α and $\beta \in R$, then $N(\alpha\beta) = N(\alpha)N(\beta)$.
- *b. Show that if $u \in R$ is a unit if and only if $N(u) = \pm 1$.
- c. Show that when $d = -1$, R has exactly 4 units.
- d. Show that when $d < -1$, R has exactly 2 units.
- e. Show that if $d = 3$ then there are infinitely many units in R . **Hint:** if u is a unit then u^k is also a unit for all $k \in \mathbb{Z}$.

Solution:

2. Let R be a ring.
 - a. Let $a \in R$. Suppose that a is not a zero divisor. Show that cancellation holds for a ; that is, show that if $ab = ac$ then $b = c$.
 - b. Show that if a is not a zero divisor and $ab = 1_R$ for some element $b \in R$, then $ba = 1_R$.

Solution:

3. Let $R = \{0, 1, a, b\}$ be a ring where a and b are units. Find the multiplication table of R . In other words find what the four ?'s should be. Give your reasons.

\cdot	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	?	?
b	0	b	?	?

Also find the addition table for this ring.

Solution:

Since a is a unit there is an element a^{-1} . So if $ax = ay$ then $x = y$. This means the row labelled a cannot have any repeated elements. The same applies to the column headed with a and also to the column headed by b . So the two ?'s in the a -row must be 1 and b in some order. But the fourth column already has a b in it so we must have b first and 1 second. Now we can easily fill in the last row since each column must have all 4 elements:

\cdot	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

For addition note that every element x has an additive inverse, namely $-x$. So each row and each column of the addition table must have all 4 elements. So $1 + 1$ must be 0, a , or b . Suppose $1 + 1 = a$. Then $a + b$ must be either b or 0. But we can't have $1 + b = b$ So $1 + b = 0$ and $1 + a$ must be b . Since addition is commutative, the table so far is

$+$	0	1	a	b
0	0	1	a	b
1	1	a	b	0
a	a	b		
b	b	0		

It is now easy to see there is only one way to fill in the addition table under the assumption $1 + 1 = a$:

$+$	0	1	a	b
0	0	1	a	b
1	1	a	b	0
a	a	b	0	1
b	b	0	1	a

But then $0 = a + a = (1 + 1)a = a^2$. Multiplying by $a^{-1} = b$ gives $0 = a$, a contradiction. So $1 + 1$ cannot be a . A similar argument shows it cannot be b . So we must have $1 + 1 = 0$. The only way to fill in the table is

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0