MATH 412 HW 4: September 28, 2015

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1. Show that every element of $\mathbb{Z}_n$ is either a unit or a zero divisor but not both.

Solution:

2. Show that $1 + 3x$ is a unit in $\mathbb{Z}_9[x]$.

Solution:

3. Let $d \in \mathbb{Z}$ be a squarefree integer which is not 0 or 1. Show that

$$R = \left\{ \begin{pmatrix} a & bd \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$$

forms a ring using matrix addition and multiplication. Show that R is isomorphic to the ring $\mathbb{Z}[\sqrt{d}]$ that was given in the previous exercice set. Show that if $d = 4$ then R is still a ring but it is not isomorphic to $\mathbb{Z}[\sqrt{4}]$ $(= \mathbb{Z}[2] = \mathbb{Z})$.

Solution:

Since R is a subset of $M_2(\mathbb{Z})$, we need only show that it is closed under addition, multiplication and negation. Suppose both $\begin{pmatrix} a & bd \\ b & a \end{pmatrix}$ and $\begin{pmatrix} a' & b'd \\ b' & a' \end{pmatrix}$ are in R. Then

$$\begin{aligned} \begin{pmatrix} a & bd \\ b & a \end{pmatrix} \begin{pmatrix} a' & b'd \\ b' & a' \end{pmatrix} &= \begin{pmatrix} aa' + bb'd & ab'd + a'bd \\ ba' + ab' & aa' + bb'd \end{pmatrix} \\ &= \begin{pmatrix} aa' + bb'd & (ab' + a'b)d \\ ba' + ab' & aa' + bb'd \end{pmatrix}. \end{aligned}$$

This last matrix has the form $\begin{pmatrix} a'' & b''d \\ b'' & a'' \end{pmatrix}$, so it is in R. Thus R is closed under multiplication. Similar (but easier) calculations show it is also closed under addition and negation.

Define a map $f : \mathbb{Z}[\sqrt{d}] \to R$ by

$$f(a + b\sqrt{d}) = \begin{pmatrix} a & bd \\ b & a \end{pmatrix}.$$

Since d is squarefree and so $\sqrt{d}$ is irrational, $a+b\sqrt{d} = a'+b'\sqrt{d}$ implies $a = a'$ and $b = b'$. Hence f is well defined. It is clearly one-to-one and straightforward calculations show it preserves the operations. For example, since

$$(a + b\sqrt{d})(a' + b'\sqrt{d}) = aa' + bb'd + (ab' + a'b)\sqrt{d}$$

we see from our formula above for products in R that f preserves multiplication.

Now suppose $d = 4$. Choosing $a = 0$ and $b = 1$, we see that

$$A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \in R.$$

Since $A^2 = 4I_2$, A is a solution to the equation $x^2 = 4$. But $2I_2$ and $-2I_2$ are also solutions so the equation has at least 3 solutions in R. But there are only two solutions in $\mathbb{Z}$. Since an isomorphism must take solutions to solutions (efts), R and $\mathbb{Z}$ are not isomorphic.