MATH 412 HW 7

BILLY BOB

1. Let I be an ideal of a ring R and let S be a subring of R. Show that $I \cap S$ is an ideal of S.

Solution:

2. Let S be the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and let I be the set of matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, where a, b and c are in $\mathbb{R}$.

a. Show that S is a ring and that I is an ideal of S.

Solution:

b. Show that $S/I \cong \mathbb{R} \times \mathbb{R}$. **Hint:** Show that $f : S \to \mathbb{R} \times \mathbb{R}$ defined by $f \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (a, c)$ is a homomorphism from S onto $\mathbb{R} \times \mathbb{R}$ with kernel I. Now invoke the first isomorphism theorem.

Solution:

3. Let p be a prime integer and let T be the set of rational numbers (in lowest terms) whose denominators are not divisible by p. Let I be the set of elements in T such that the numerator is divisible by p.

a. Prove T is a ring and that I is an ideal of T.

Solution:

b. Show that T/I is isomorphic to $\mathbb{Z}_p$. **Hint:** See the hint for problem 2.

Solution: Let $a/b \in T$, where a and $b \in \mathbb{Z}$ and $\gcd(b, p) = 1$. For $a \in \mathbb{Z}$ let $[a]_p$ denote the equivalence class of a modulo p. Define a map $f : T \twoheadrightarrow \mathbb{Z}_p$ by

$$f(a/b) = [a]_p \cdot ([b]_p)^{-1}$$

Notice that since b is relatively prime to p it is invertible in $\mathbb{Z}_p$; so the above definition makes sense. Also if a/b and a'/b' are equal as rational numbers and both b and b' are relatively prime to p, then $f(a/b) = f(a'/b')$. We leave the proof of this to the reader. We claim f is a homomorphism. To see it preserves products we calculate

$$\begin{aligned}
f(\frac{a}{b}\cdot\frac{c}{d}) &= f(\frac{ac}{bd}) \\
&= [ac]_p([bd]_p)^{-1} \\
&= [a]_p[c]_p([b]_p[d]_p)^{-1} \\
&= [a]_p[c]_p([b]_p)^{-1}([d]_p)^{-1} \\
&= [a]_p([b]_p)^{-1}[c]_p([d]_p)^{-1} \\
&= f(\frac{a}{b})f(\frac{c}{d})
\end{aligned}$$

A similar calculation shows that f preserves subtraction and thus f is a homomorpism. Clearly f is onto.
From the definitions of f we see that $f(a/b) = [0]_p$ if and only if $[a]_p = [0]_p$. And this happens if and only if a is a multiple of p, which happens if and only if $a/b \in I$. So I is the kernel of f and by the first isomorphism theorem, $T/I \cong \mathbb{Z}_p$.

Extra Credit Problem

4. Find the lattice of ideals of the ring T from problem 3. There are infinitely many ideals but the lattice is pretty simple.

Solution: