Math 412 Final, Take Home Part
Due Dec 13, 2012

This is open book.

1. Let $R$ be the ring of all upper triangular $n \times n$ matrices with entries in a field. Let $I$ be the set of all strictly upper triangular matrices. ($A$ is upper triangular if $a_{ij} = 0$ whenever $i > j$; it is strictly upper triangular if $a_{ij} = 0$ whenever $i \geq j$.)

   a. Show $I$ is an ideal of $R$.
   Solution:

   b. Show $R/I \cong S$, where $S$ is the ring of diagonal matrices.
   Solution:

2. Let $R$ be a ring in which $x^2 = x$ holds for every $x \in R$.

   a. Show that $2a = 0$ for all $a \in R$. (Of course $2a$ means $a + a$.)
   Solution:

   b. Show $R$ is commutative.
   Solution:

3. Find all monic irreducible polynomials of

   a. degree 2 in $\mathbb{Z}_3[x]$.
   Solution:

   b. degree 3 in $\mathbb{Z}_2[x]$.
   Solution:

4. Determine if the given polynomial is irreducible:

   a. $x^3 - 9$ in $\mathbb{Z}_{11}[x]$  
   Solution:

   b. $x^4 + x^2 + 1$ in $\mathbb{Z}_3[x]$  
   Solution:

   c. $x^4 + 2x^2 + 2x + 2$ in $\mathbb{Z}_3[x]$  
   Solution:
5. Show that there are infinitely many integers $k$ such that $x^9 + 12x^5 - 21x + k$ is irreducible in $\mathbb{Q}[x]$. Hint: Use Eisenstein.

Solution:

6. An element $a$ of a ring $R$ of a nilpotent if $a^n = 0$ for some $n$. Assume $R$ is commutative.

   a. If $a^n = 0$ and $b^m = 0$ in $R$, show that $(a + b)^{n+m-1} = 0$. Hint: the binomial theorem, which is true in commutative rings, says

   $$(a + b)^k = \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}$$

   Solution:

   b. Show that the set $N$ of all nilpotent elements of $R$ is an ideal and that $R/N$ has no nonzero nilpotent element.

   Solution:

   c. Show that in $M_2(F)$, $F$ a field, the set of nilpotent elements do not form an ideal. Hint: Find $A, B \in M_2(F)$ which satisfy $A^2 = 0$ and $B^2 = 0$ but $A + B$ is not nilpotent.

   Solution: