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AN EASY TEST FOR CONGRUENCE MODULARITY

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Abstract. We describe an easy way to determine whether the realization of a set of idempotent identities guarantees congruence modularity or the satisfaction of a nontrivial congruence identity. Our results yield slight strengthenings of Day's Theorem and Gumm's Theorem, which each characterize congruence modularity.

1. Introduction

Given a set Σ of identities, how does one determine whether the variety axiomatized by Σ is congruence modular? One natural approach is to look for Day terms (see [4] or Theorem 3.1 below). In this paper we will exhibit an easier method, which works when Σ is a set of *idempotent* identities. By this, we mean that for every function symbol F appearing in Σ, it is the case that $\Sigma \models F(x, \dots, x) \approx x$.

For a set Σ of idempotent identities we shall define the notion of a *derivative*, Σ', which is a superset of idempotent identities in the same language. One of our main theorems is that Σ axiomatizes a congruence modular variety if the derivative of Σ is inconsistent. Another main theorem is that Σ axiomatizes a variety that satisfies some nontrivial congruence identity if its n-th derivative is inconsistent for some n. (In fact, we prove our results not only for the variety $\mathcal{V}$ axiomatized by Σ, but also any variety that "interprets $\mathcal{V}$".) Our final two theorems show that for a set Σ of idempotent linear identities, the derivative test is a necessary and sufficient condition to determine if Σ defines a variety that is congruence modular or satisfies a nontrivial congruence identity.

2. Definitions

Let Σ be a set of identities. Σ is *inconsistent* if $\Sigma \models x \approx y$, otherwise Σ is *consistent*. Σ is *idempotent* if for every function symbol F appearing in Σ it is the case that $\Sigma \models F(x, x, \dots, x) \approx x$. F is *weakly independent of its first place* if $\Sigma \models F(y, \mathbf{w}) \approx x$ for variables $x \neq y$ and some sequence of not necessarily distinct variables $\mathbf{w}$. F is *independent of its first place* if $\Sigma \models F(x, \mathbf{z}) \approx F(y, \mathbf{z})$, where x, y, and all variables in the sequence $\mathbf{z}$ are distinct. Define independence and weak independence of each of the other places in the same way. (If Σ is idempotent and

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F is independent of its first place, then F is weakly independent of its first place. This is because the assumptions that Σ is idempotent and F is independent of its first place yield $\Sigma \models F(y, x, x, \ldots, x) \approx F(x, x, x, \ldots, x) \approx x$, which suffices to show that F is weakly independent of its first place.) The concepts of weak independence and ordinary independence are defined relative to Σ, so when it is not obvious we will specify explicitly which set Σ is involved.

Σ is *realized* by an algebra $\mathbf{A}$ (or variety $\mathcal{V}$) if it is possible to interpret each function symbol appearing in Σ as a term of $\mathbf{A}$ (respectively $\mathcal{V}$) such that all identities in Σ are satisfied by $\mathbf{A}$ (respectively $\mathcal{V}$).

Let Σ be a set of idempotent identities, and let $\mathcal{P}$ be the set of pairs (F, i) where F is a function symbol appearing in Σ that is weakly independent of its i-th place. The *derivative* Σ' of Σ is the set of identities obtained by adding to Σ all identities asserting that F is independent of its i-th place for all pairs $(F, i) \in \mathcal{P}$. (I.e., Σ' strengthens each instance of weak independence to an instance of independence.) The n-th derivative of Σ is denoted $\Sigma^{(n)}$.

For example, if Σ is the set consisting of the two identities (i) $F(y, y, x) \approx x$ and (ii) $F(x, y, y) \approx x$, then from (i) we derive that F is weakly independent of its first and second place, while from (ii) we derive that F is weakly independent of its second and third place. Hence Σ' will contain identities (iii) $F(x, z_2, z_3) \approx F(y, z_2, z_3)$, (iv) $F(z_1, x, z_3) \approx F(z_1, y, z_3)$ and (v) $F(z_1, z_2, x) \approx F(z_1, z_2, y)$, which assert that F is independent of all of its places. In this example, Σ' is inconsistent, since $\Sigma' \models x \approx F(x, y, y) \approx F(y, y, y) \approx y$, where the first instance of $\approx$ is from Σ, the second follows from (iii) by variable replacement, and the third follows from (i) by variable replacement.

3. Testing for congruence modularity

In the introduction we raised the question of how to determine whether the variety axiomatized by a set of identities Σ is congruence modular. Rather than consider the variety axiomatized by Σ we shall consider varieties that realize Σ, since this is more general. (The variety axiomatized by Σ also realizes Σ, since we may interpret each function symbol appearing in Σ as itself.)

We start with Alan Day's characterization of congruence modularity.

Theorem 3.1. [4] *The following are equivalent for a variety $\mathcal{V}$.*

1. *$\mathcal{V}$ is congruence modular.*
2. *There exist 4-variable terms $m_0, \ldots, m_n$ such that the following identities hold in $\mathcal{V}$:*
   - (a) *$m_0(x, u, v, y) \approx x$ and $m_n(x, u, v, y) \approx y$,*
   - (b) *$m_i(x, y, y, x) \approx x$,*
   - (c) *$m_i(x, u, u, y) \approx m_{i+1}(x, u, u, y)$ for i odd, and*
   - (d) *$m_i(x, x, y, y) \approx m_{i+1}(x, x, y, y)$ for i even.*

From this we derive our first main result.

Theorem 3.2. *$\mathcal{V}$ is congruence modular if and only if $\mathcal{V}$ realizes some set Σ of idempotent identities whose derivative is inconsistent.*

Proof. [$\Rightarrow$] Assume that $\mathcal{V}$ is congruence modular, and that Σ is the set of identities guaranteed by Theorem 3.1. The identities in part (2)(b) suffice to guarantee that this is a set of idempotent identities. Moreover, the identities of type (2)(b) suffice to guarantee that Σ' will contain identities expressing that each $m_i(x, u, v, y)$ is independent of its middle two places. In light of this, the identities of type (2)(c) and (2)(d) together assert that $\Sigma' \models m_i(x, *, *, y) \approx m_{i+1}(x, *, *, y)$ for all i, where the asterisks indicate that the identity holds for any middle values. This and (2)(a) yield

$$\Sigma' \models x \approx m_0(x, *, *, y) \approx m_1(x, *, *, y) \approx \cdots \approx m_n(x, *, *, y) \approx y,$$

so Σ' is inconsistent.

[$\Leftarrow$] Conversely, assume that $\mathcal{V}$ is a variety realizing Σ that is not congruence modular. We need to prove that Σ' is consistent. Without loss of generality we may assume that $\mathcal{V}$ is the variety axiomatized by Σ. For, if $\mathcal{V}$ contains an algebra $\mathbf{A}$ whose congruence lattice is nonmodular, then the reduct of $\mathbf{A}$ to the symbols in Σ is an algebra in the variety axiomatized by Σ whose congruence lattice is nonmodular.

Now we follow Day's proof of Theorem 3.1, and show how to extract a nontrivial model of Σ' from a failure of congruence modularity. Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(a, b, c, d)$ be a 4-generated $\mathcal{V}$-free algebra. Define congruences

$$\alpha = \theta((a, b), (c, d)),\ \beta = \theta((a, d), (b, c)), \text{ and } \gamma = \theta(b, c).$$

Day's theorem proves that $\mathcal{V}$ is congruence modular if and only if

$$\beta = (\alpha \wedge \beta) \vee \gamma. \tag{3.1}$$

Since $\mathcal{V}$ is idempotent, we can simplify the situation. Let $\mathbf{T} = \mathbf{F}_{\mathcal{V}}(r, s)$ be a 2-generated $\mathcal{V}$-free algebra. The congruences α and β are the kernels of the homomorphisms $A, B\colon \mathbf{F} \to \mathbf{T}$ defined by $A\colon a, b \mapsto r; c, d \mapsto s$ and $B\colon a, d \mapsto r; b, c \mapsto s$. Hence $\alpha \wedge \beta$ is the kernel of the homomorphism

$$A \times B\colon \mathbf{F} \to \mathbf{T}^2\colon a \mapsto (r, r);\ b \mapsto (r, s);\ c \mapsto (s, s);\ d \mapsto (s, r).$$

$A \times B$ is surjective, since if $p = p(r, s)$ and $q = q(r, s) \in T$ are arbitrarily chosen, then $m = p(q(a, b), q(d, c)) \in F$ is an element such that $(A \times B)(m) = (p, q)$. The kernel of $A \times B$ is $\alpha \wedge \beta$, so

$$\mathbf{F}/(\alpha \wedge \beta) \cong \mathbf{T}^2. \tag{3.2}$$

Since both sides of (3.1) contain $\alpha \wedge \beta$, we can express Day's conclusion in terms of the algebra $\mathbf{T}^2$. If η_1 and η_2 are the coordinate projection kernels of $\mathbf{T}^2$, then η_1 corresponds to $\alpha/(\alpha \wedge \beta)$ and η_2 corresponds to $\beta/(\alpha \wedge \beta)$ under the isomorphism (3.2), and $\delta = \theta((r, s), (s, s))$ corresponds to $\gamma/(\alpha \wedge \beta) = ((\alpha \wedge \beta) \vee \gamma)/(\alpha \wedge \beta)$.

Day's conclusion is that $\mathcal{V}$ is congruence modular if and only if $\eta_2 = \delta$. Since we are assuming that $\mathcal{V}$ is not congruence modular, and since

$$\eta_2 = \theta\big(((r,r),(s,r)),\ ((r,s),(s,s))\big) \supseteq \theta((r,s),(s,s)) = \delta,$$

it follows that $((r,r),(s,r)) \in \eta_2 \setminus \delta$.

Since $\mathcal{V}$ is idempotent, each congruence class is a subuniverse of $\mathbf{T}^2$. Let $\mathbf{G} \leq \mathbf{T}^2$ be the subalgebra whose universe is the η_2-class of (r,r). We argue that the nontrivial quotient algebra $\overline{\mathbf{G}} = \mathbf{G}/\delta|_G$ is a model of Σ'. $\overline{\mathbf{G}}$ is a model of Σ because it is a section of $\mathbf{T}^2 \in \mathcal{V}$ and $\mathcal{V}$ is the class of all models of Σ.

Let F be an n-place function symbol appearing in Σ, and suppose that

$$\Sigma \models F(y, \mathbf{w}) \approx x \tag{3.3}$$

where $x \neq y$ and $\mathbf{w}$ is a sequence of not necessarily distinct variables. Let $U \subseteq \{1,\dots,n\}$ be the set of places of F where x occurs in this identity.

Claim 3.3. *Choose any $(g_1,r),\dots,(g_n,r) \in G$. Define*

$$r_i = \begin{cases} r & \text{if } i \in U; \\ s & \text{else.} \end{cases}$$

Then

$$F^{\mathbf{T}^2}((g_1,r),\dots,(g_n,r)) = F^{\mathbf{T}^2}((g_1,r_1),\dots,(g_n,r_n)). \tag{3.4}$$

That both sides of (3.4) are equal in the first coordinate is trivial. In the second coordinate we must establish that the value on the left-hand side, which is $F^{\mathbf{T}}(r,\dots,r) = r$, is the same as the value $F^{\mathbf{T}}(r_1,\dots,r_n)$ on the right-hand side, which is obtained by evaluating $F^{\mathbf{T}}$ on an $\{r,s\}$-tuple with r's substituted in position i for each $i \in U$ and s's substituted in all other positions. This follows from (3.3).

Claim 3.4. *For any $w \in T$, $((w,s),(s,s)) \in \delta$.*

If $w(x,y)$ is a binary term such that $w = w(r,s)$, then

$$(w,s) = (w(r,s), w(s,s)) = w((r,s),(s,s)) \equiv_\delta w((s,s),(s,s)) = (s,s).$$

Now we prove that F is independent of its first place modulo δ on $\mathbf{G}$. Choose $(g_1,r),\dots,(g_n,r),(h,r) \in G$ arbitrarily. By Claim 3.3,

$$F^{\mathbf{T}^2}((g_1,r),(g_2,r),\dots,(g_n,r)) = F^{\mathbf{T}^2}((g_1,r_1),(g_2,r_2),\dots,(g_n,r_n)),$$

and similarly

$$F^{\mathbf{T}^2}((h,r),(g_2,r),\dots,(g_n,r)) = F^{\mathbf{T}^2}((h,r_1),(g_2,r_2),\dots,(g_n,r_n)).$$

Now $r_1 = s$, since $1 \notin U$ according to (3.3), and by Claim 3.4 we have $(g_1, s) \equiv_\delta (h, s)$, therefore

$$\begin{aligned} F^{\mathbf{T}^2}((g_1, r), (g_2, r), \ldots, (g_n, r)) &= F^{\mathbf{T}^2}((g_1, s), (g_2, r_2), \ldots, (g_n, r_n)) \\ &\equiv_\delta F^{\mathbf{T}^2}((h, s), (g_2, r_2), \ldots, (g_n, r_2)) \\ &= F^{\mathbf{T}^2}((h, r), (g_2, r), \ldots, (g_n, r)). \end{aligned}$$

This proves that F is independent of its first place modulo δ on G, so

$$\mathbf{G}/\delta|_{\mathbf{G}} \models F(x, \mathbf{z}) \approx F(y, \mathbf{z})$$

where x, y, and all variables in the sequence $\mathbf{z}$ are distinct. □

Corollary 3.5. *Day's Theorem (Theorem 3.1) remains true if one weakens*

(c) $m_i(x, u, u, y) \approx m_{i+1}(x, u, u, y)$ *for* i *odd,*

to either

(c)′ $m_i(x, x, x, y) \approx m_{i+1}(x, x, x, y)$ *for* i *odd, or*
(c)″ $m_i(x, y, y, y) \approx m_{i+1}(x, y, y, y)$ *for* i *odd.*

In particular, congruence modularity can be characterized by identities involving only the variables x *and* y*.*

Proof. We can obtain (c)′ and (c)″ from (c) by replacing the variable u by either x or y, so (c)′ and (c)″ are formally weaker than (c). If you take Σ to be the set of identities of Theorem 3.1 with (c) replaced by either (c)′ or (c)″, then Σ' is inconsistent by the same argument used in the proof of direction [⇒] of Theorem 3.2. Thus, the weakened identities still imply congruence modularity.

For the last statement of the corollary, we can delete the terms m_0 and m_n from the list of Day terms and just use x and y in their place. Then, with (c) replaced by either (c)′ or (c)″, the identities involve only x and y. □

We believe that the first published proof that congruence modularity can be characterized by 2-variable identities appears in [24] by J. B. Nation (see the corollary on page 85 of that paper). Nation's 2-variable identities use 5-variable terms.

Example 3.6. A lattice is *p-modular* if it satisfies the identity

$$(x \vee (y \wedge z)) \wedge (z \vee (y \wedge x)) = (z \wedge (x \vee (y \wedge z))) \vee (x \wedge (z \vee (y \wedge x))). \tag{3.5}$$

This identity is satisfied by all modular lattices and some nonmodular lattices. (It is the conjugate identity for a 10-element splitting lattice.)

Eva Gedeonová characterized the satisfaction of (3.5) as a congruence identity with the following theorem.

Theorem 3.7. [7] *The following are equivalent for a variety* $\mathcal{V}$*.*

(1) $\mathcal{V}$ *satisfies (3.5) as a congruence identity.*
(2) *There exist 6-variable terms* $g_0, \ldots, g_n$ *such that the following identities hold in* $\mathcal{V}$*:*

(a) $g_0(x,s,t,u,v,y) \approx x$ *and* $g_n(x,s,t,u,v,y) \approx y$,
(b) $g_i(x,x,y,y,x,x) \approx g_i(x,y,x,x,y,x) \approx x$ *for all* i,
(c) $g_i(x,s,x,y,s,y) \approx g_{i+1}(x,s,x,y,s,y)$ *for* i *odd, and*
(d) $g_i(x,x,s,s,y,y) \approx g_{i+1}(x,x,s,s,y,y)$ *for* i *even.*

Although the p-modular law is strictly weaker than the modular law as a lattice identity, Day was able to show in [5] that any variety realizing the set Σ of identities of Theorem 3.7 (2) is congruence modular. His argument involved nonobvious calculations with the congruences of the 4-generated free algebra in a variety realizing these identities.

We will derive Day's result from our Theorem 3.2. Gedeonová's identity (b) implies that Σ is idempotent, hence our theorem applies. Identity (b) also implies that each g_i is weakly independent of its middle four places relative to Σ. Hence

$$\Sigma' \models x \overset{\text{(a)}}{\approx} g_0(x,*,*,*,*,y) \overset{\text{(d)}}{\approx} g_1(x,*,*,*,*,y) \overset{\text{(c)}}{\approx} \cdots \approx g_n(x,*,*,*,*,y) \overset{\text{(a)}}{\approx} y.$$

Σ' is inconsistent, so any congruence p-modular variety is congruence modular.

Example 3.8. The paper [2] introduces the concept of a "cube term", which is a common generalization of a Maltsev term and a near unanimity term. A cube term is a term $F(x_1,\dots,x_n)$, for some $n \geq 3$, satisfying an idempotent set of identities Σ which expresses exactly that F is weakly independent of each of its places.

It is proved in [2] that a variety with a cube term (i.e., a variety realizing Σ) is congruence modular. The method of proof is to show first that a variety with a cube term has an "edge term", and then that a variety with an edge term has Day terms. The first step of the proof is long ($\approx$ 5 journal pages) and highly nontrivial. The second step is short and easy to verify, but it is easy to imagine that it required ingenuity to discover.

We can prove the combination of both steps with no ingenuity. Since Σ asserts that F is weakly independent of all places, Σ' expresses that F is constant. At the same time, Σ' expresses that F is idempotent (since $\Sigma' \supseteq \Sigma$). Thus Σ' proves that $F(x,x,\dots,x)$ interprets simultaneously as a constant function and as the identity function on any algebra realizing Σ. It follows that Σ' has no models of size greater than one. Hence Theorem 3.2 applies, showing that any variety realizing Σ is congruence modular.

Example 3.9. A variety is congruence n-permutable if, for any two congruences α and β on any algebra $\mathbf{A} \in \mathcal{V}$, it is the case that the n-fold relational product $\alpha \circ_n \beta = \alpha \circ \beta \circ \alpha \circ \beta \circ \cdots$ equals $\beta \circ_n \alpha$. This property was characterized by Joachim Hagemann and Aleit Mitschke in the following way.

Theorem 3.10. [9] *The following are equivalent for a variety* $\mathcal{V}$.

(1) $\mathcal{V}$ *is congruence* n*-permutable.*

(2) *There exist* 3*-variable terms* $p_0, \dots, p_n$ *such that the following identities hold in* $\mathcal{V}$*:*
   (a) $p_0(x,u,y) \approx x$ *and* $p_n(x,u,y) \approx y$,
   (b) $p_i(x,x,y) \approx p_{i+1}(x,y,y)$ *for all* i.

Bjarni Jónsson proved in [11] that any congruence 3-permutable variety is congruence modular. On the other hand, there exist congruence 4-permutable varieties that are not congruence modular. The first of these statements can be proved by a computation, but we derive it from Theorem 3.2.

A congruence 3-permutable variety realizes

$$\Sigma = \{x \approx p_1(x,y,y),\ p_1(x,x,y) \approx p_2(x,y,y),\ p_2(x,x,y) \approx y\}.$$

Σ' contains Σ along with (i) identities asserting that p_1 is independent of its second and third places (from the first identity of Σ) and (ii) identities asserting that p_2 is independent of its first and second places (from the third identity of Σ). Hence

$$\Sigma' \models x \overset{\Sigma}{\approx} p_1(x,y,y) \overset{\text{(i)}}{\approx} p_1(x,x,y) \overset{\Sigma}{\approx} p_2(x,y,y) \overset{\text{(ii)}}{\approx} p_2(x,x,y) \overset{\Sigma}{\approx} y,$$

showing that Σ' is inconsistent.

The derivative test can also be used to prove that congruence 4-permutability does *not* imply congruence modularity. The method for proving this is explained in Section 5.

Example 3.11. In [1], Wolfram Bentz investigated varieties $\mathcal{V}$ whose T_0 topological algebras are Hausdorff. It is conjectured that these varieties are exactly the congruence modular varieties that are congruence n-permutable for some n. (This conjecture, called "the Congruence Modularity Conjecture", is still open.)

Let Σ_1 be the set consisting of the following identities involving the 3-variable terms q_1, q_2, p:

(a) $x \approx q_1(x,y,y)$,
(b) $q_1(x,x,y) \approx q_2(x,x,y)$,
(c) $q_2(x,y,x) \approx x$ and $q_2(x,y,y) \approx p(x,y,y)$,
(d) $p(x,x,y) \approx y$.

(These are the Gumm identities for congruence modularity, from [8], for $n = 2$ *minus* the Gumm identity $q_1(x,y,x) \approx x$.) Now let $\mathcal{P}_n$ be the set of identities listed in Theorem 3.10 (2). Bentz proved that any variety realizing $\Sigma_1 \cup \mathcal{P}_n$, for any given n, has the property that its T_0 topological algebras are Hausdorff. In light of the Congruence Modularity Conjecture, this led him to raise the question of whether varieties realizing $\Sigma_1 \cup \mathcal{P}_n$ must be congruence modular.

The question raised by Bentz was answered by Kearnes and Luís Sequeira in [18], where it was shown that any variety realizing Σ_1 must already be congruence modular.

The proof was accomplished by defining

$$\begin{aligned}
m_0(x,u,v,y) &:= x,\\
m_1(x,u,v,y) &:= x,\\
m_2(x,u,v,y) &:= q_1(x,q_2(x,v,u),p(x,u,v)),\\
m_3(x,u,v,y) &:= q_2(x,u,y),\\
m_4(x,u,v,y) &:= q_2(x,v,y),\\
m_5(x,u,v,y) &:= p(u,v,y),\\
m_6(x,u,v,y) &:= y,
\end{aligned}$$

and showing that it is provable from Σ_1 that the Day identities of Theorem 3.1 hold for these new terms.

Here we give a different proof that any variety realizing Σ_1 is congruence modular, based on Theorem 3.2. Identities (a), (c) and (d) from the definition of Σ_1 suffice to prove that Σ_1 is idempotent, so the theorem applies. Identities (a), (c), and (d) also show that q_1, q_2, and p are weakly independent of their middle places relative to Σ_1. Therefore, q_1, q_2, and p are independent of their middle places relative to Σ_1'. Hence

$$\Sigma_1' \models x \overset{\text{(a)}}{\approx} q_1(x,*,y) \overset{\text{(b)}}{\approx} q_2(x,*,y) \overset{\text{(c)}}{\approx} p(x,*,y) \overset{\text{(d)}}{\approx} y,$$

showing that Σ_1' is inconsistent.

The argument from Example 3.11 actually establishes the following theorem, a slight strengthening of H. Peter Gumm's Theorem from [8].

Theorem 3.12. *The following are equivalent for a variety $\mathcal{V}$.*

(1) *$\mathcal{V}$ is congruence modular.*
(2) *There exist 3-variable terms $q_0, \dots, q_n, p$ such that the following identities hold in $\mathcal{V}$:*
   (a) $q_0(x,u,y) \approx x$,
   (b) $q_i(x,y,x) \approx x$ *for i in the interval* $[2,n]$,
   (c) $q_i(x,y,y) \approx q_{i+1}(x,y,y)$ *for i even,*
   (d) $q_i(x,x,y) \approx q_{i+1}(x,x,y)$ *for i odd,*
   (e) $q_n(x,y,y) \approx p(x,y,y)$, *and*
   (f) $p(x,x,y) \approx y$.

If Σ is the set of identities in Theorem 3.12, then these are exactly Gumm's identities from [8] *minus* the identity $\varepsilon : q_1(x,y,x) \approx x$. The theorem asserts that we can delete this single identity from Gumm's set and still have a set of identities forcing congruence modularity. The 'reason' for this is that the only role played by this identity ε in our method is to prove that q_1 is weakly independent of its middle place

relative to the Gumm identities $\Sigma \cup \{\varepsilon\}$. But this can be deduced another way, directly from Σ, using $\Sigma \models x \overset{(a)}{\approx} q_0(x,y,y) \overset{(c)}{\approx} q_1(x,y,y)$.

Example 3.13. Theorem 3.2 proves that for arbitrary set Σ of idempotent identities, if Σ' is inconsistent, then every variety realizing Σ is congruence modular. The converse is false for some Σ, as the following example shows.

Let Σ be a set of identities that axiomatizes the variety of lattices. Thus Σ is a set of idempotent identities. The only function symbols appearing in Σ are $\wedge$ and $\vee$, which are not weakly independent of any of their two places relative to Σ, therefore $\Sigma' = \Sigma$ is consistent. However, any variety realizing Σ, i.e., any variety of expanded lattices, is congruence distributive, hence congruence modular.

4. Testing for a nontrivial congruence identity

Our result about congruence modular varieties has an analogue for varieties that satisfy a nontrivial congruence identity. First we recall one Maltsev characterization for this class of varieties.

Theorem 4.1. [15] *The following are equivalent for a variety $\mathcal{V}$.*

1. *$\mathcal{V}$ satisfies a nontrivial congruence identity.*
2. *There exist 4-variable terms $M_0, \dots, M_n$ such that the following identities hold in $\mathcal{V}$:*
   - (a) *$M_0(x,u,v,y) \approx x$ and $M_n(x,u,v,y) \approx y$,*
   - (b) *$M_i(x,x,y,y) \approx M_{i+1}(x,x,y,y)$ and $M_i(x,y,x,y) \approx M_{i+1}(x,y,x,y)$ for i odd, and*
   - (c) *$M_i(x,y,y,y) \approx M_{i+1}(x,y,y,y)$ for i even.*

There are other Maltsev characterizations of the class of varieties satisfying nontrivial congruence identities given in Definition 2.17, Theorem 5.23 and Theorem 8.13 of [15], which could be used just as easily in this paper. We chose the one above, which is part of Theorem 5.28 of [15], since it is the characterization that most resembles the characterization of congruence modularity in Theorem 3.1.

Theorem 4.2. *$\mathcal{V}$ satisfies a nontrivial congruence identity if and only if $\mathcal{V}$ realizes some set Σ of idempotent identities whose n-th derivative is inconsistent for some n.*

Proof. [$\Rightarrow$] Assume that $\mathcal{V}$ satisfies a nontrivial congruence identity, and that Σ is the set of identities guaranteed by Theorem 4.1. The consequence of these identities that results from replacing all variables with x is

$$\Sigma \models x \approx M_0(x,x,x,x) \approx M_1(x,x,x,x) \approx \cdots \approx M_n(x,x,x,x),$$

showing that Σ is idempotent.

Claim 4.3. *$\Sigma^{(i)} \models M_i(x,u,v,y) \approx x$ for all i.*

The claim holds for $i = 0$ by identity (2)(a) of Theorem 4.1. Let's assume that the claim holds for some $k \geq 0$, and prove it for $k+1$. If k is odd, then from (2)(b) of Theorem 4.1 we derive

$$\begin{aligned} \Sigma^{(k)} &\models \quad x \approx M_k(x,x,y,y) \approx M_{k+1}(x,x,y,y) \qquad \text{and} \\ \Sigma^{(k)} &\models \quad x \approx M_k(x,y,x,y) \approx M_{k+1}(x,y,x,y), \end{aligned}$$

from which we conclude that $M_{k+1}(x,u,v,y)$ is weakly independent of its second, third and fourth places relative to $\Sigma^{(k)}$. Hence

$$\Sigma^{(k+1)} \models M_{k+1}(x,u,v,y) \approx M_{k+1}(x,x,x,x) \approx x, \tag{4.1}$$

as claimed. If k is even, then from (2)(c) of Theorem 4.1 we derive

$$\Sigma^{(k)} \models \quad x \approx M_k(x,y,y,y) \approx M_{k+1}(x,y,y,y),$$

from which we conclude that $M_{k+1}(x,u,v,y)$ is weakly independent of its second, third and fourth places relative to $\Sigma^{(k)}$. Just as in (4.1), this finishes the claim in the case where k is even.

It follows from the claim and identity (2)(a) of Theorem 4.1 that $\Sigma^{(n)} \models x \approx M_n(x,u,v,y) \approx y$, so $\Sigma^{(n)}$ is inconsistent. This concludes the proof of [$\Rightarrow$].[1]

[$\Leftarrow$] We will argue that if Σ is idempotent and the realization of Σ by $\mathcal{V}$ does not force $\mathcal{V}$ to satisfy a nontrivial congruence identity, then the same properties are true for Σ'.

If Σ is idempotent, then so is Σ', since it extends Σ and it involves no new function symbols.

If the realization of Σ by $\mathcal{V}$ does not force a nontrivial congruence identity, then the variety $\mathcal{V}_\Sigma$ axiomatized by Σ does not satisfy a nontrivial congruence identity. The combination of Theorems 2.16 and 7.15 (1)$\Leftrightarrow$(2) of [15] implies that $\mathcal{V}_\Sigma$ has no "Hobby-McKenzie term". In this situation, the contrapositive of Lemma 2.5 of [13] proves that $\mathcal{V}_\Sigma$ has a subvariety term equivalent to the variety of sets or the variety of semilattices. In either case, this means that Σ can be realized by a 2-element meet semilattice, $\mathbf{S} = \langle \{0,1\}; \wedge \rangle$. If F is weakly independent of its first place relative to Σ, then $\Sigma \models F(y, \mathbf{w}) \approx x$ for $y \neq x$ and some sequence of not necessarily distinct variables $\mathbf{w}$. By setting $x = 1$ and $u = 0$ for all other variables u occurring in $y\mathbf{w}$, if $\mathbf{s}$ denotes the sequence of elements of $\mathbf{S}$ corresponding to $\mathbf{w}$, then we obtain that $F^{\mathbf{S}}(0, \mathbf{s}) = 1$, where $F^{\mathbf{S}}$ is a semilattice term (equivalent to a meet of variables). Necessarily $F^{\mathbf{S}}$ does not depend on its first place. Thus $\mathbf{S} \models F(x, \mathbf{z}) \approx F(y, \mathbf{z})$ where x, y, and all variables in the sequence $\mathbf{z}$ are distinct. This shows that $\mathbf{S}$ is a model of Σ'. This is enough to prove that the realization of Σ' also does not force the satisfaction of a nontrivial congruence identity (since the variety of semilattices does not satisfy a nontrivial congruence identity, [6]).

[1] In fact, it is possible to show that $\Sigma^{(\lceil n/2 \rceil)}$ is inconsistent by working inward from both ends of the sequence $M_0, \ldots, M_n$ at the same time, but this does not add anything useful to this proof.

We have shown that if Σ is idempotent and the realization of Σ does not force the satisfaction of a nontrivial congruence identity, then both properties hold for Σ', hence for $\Sigma^{(n)}$ for any n. In particular, $\Sigma^{(n)}$ is consistent for any n. This is the contrapositive of $[\Leftarrow]$. □

Example 4.4. Lemma 3.10 of [12], which is credited to Day, proves that a congruence n-permutable variety with a semilattice operation satisfies a nontrivial congruence identity. In other words, if Σ_1 is the set of identities from Theorem 3.10 (2) and Σ_2 is the set of identities expressing that some binary term $s(x, y)$ is a semilattice operation, then any variety realizing the (idempotent) set of identities $\Sigma_1 \cup \Sigma_2$ satisfies a nontrivial congruence identity.

Later, in Theorem 9.19 of [10], David Hobby and Ralph McKenzie proved that any locally finite congruence n-permutable variety satisfies a nontrivial congruence identity. (No assumption is made about the existence of a semilattice term.)

The full result, that any congruence n-permutable variety satisfies a nontrivial congruence identity, was established by Paolo Lipparini in [21]. He went on to publish alternative proofs of this theorem in [20] and [22].

Another proof that any congruence n-permutable variety satisfies a nontrivial congruence identity was found by Kearnes and Nation in [17].

Here we show how to derive this theorem from Theorem 4.2. Let Σ denote the set of identities in Theorem 3.10 (2).

Claim 4.5. $\Sigma^{(i)} \models p_i(x, u, y) \approx x$ *for all* i.

The claim holds for $i = 0$ by identity (2)(a) of Theorem 3.10. Assume that the claim holds for some $k \geq 0$. From this, by identity (2)(b) of Theorem 3.10, we derive that $\Sigma^{(k)} \models x \approx p_k(x, x, y) \approx p_{k+1}(x, y, y)$, so p_{k+1} is weakly independent of its last two places relative to $\Sigma^{(k)}$. Thus p_{k+1} is fully independent of its last two places relative to $\Sigma^{(k+1)}$, and this means that $\Sigma^{(k+1)} \models p_{k+1}(x, u, y) \approx p_{k+1}(x, x, x) \approx x$. This proves the claim.

Combining the claim with Theorem 3.10 (2)(a), we get that

$$\Sigma^{(n)} \models x \approx p_n(x, u, y) \approx y,$$

hence $\Sigma^{(n)}$ is inconsistent. Now apply Theorem 4.2.

Example 4.6. Hobby and McKenzie show in Theorem 9.11 of [10] that a locally finite variety $\mathcal{V}$ has the property that its finite members have join semidistributive congruence lattices if and only if $\mathcal{V}$ has ternary terms $d_0, \ldots, d_n$ such that the following identities are satisfied in $\mathcal{V}$:

(a) $d_0(x, y, z) \approx x$ and $d_n(x, y, z) \approx z$,
(b) $d_i(x, y, y) \approx d_{i+1}(x, y, y)$ and $d_i(x, y, x) \approx d_{i+1}(x, y, x)$ for even i, and
(c) $d_i(x, x, y) \approx d_{i+1}(x, x, y)$ for odd i.

Kearnes showed in Theorem 2.6 of [14] that these locally finite varieties satisfy a nontrivial congruence identity (which depends on the number of d_i's).

Kearnes and Emil W. Kiss later showed in Theorem 8.14 of [15] that an arbitrary variety is congruence join semidistributive if and only if it has ternary terms $d_0, \dots, d_n$ for which the identities (a)–(c) are satisfied, and that these varieties satisfy a nontrivial congruence identity. The fact that a variety satisfying (a)–(c) satisfies a nontrivial congruence identity can be proved with Theorem 4.2. For this, let Σ be the set of (idempotent) identities from (a)–(c).

Claim 4.7. $\Sigma^{(n)}$ *is inconsistent.*

We argue by induction on k that (d) $\Sigma^{(k)} \models d_{n-k}(x,y,z) \approx z$. Then, when $k = n$, we get inconsistency: $\Sigma^{(n)} \models x \overset{\text{(a)}}{\approx} d_0(x,y,z) \overset{\text{(d)}}{\approx} z$.

Assertion (d) holds for $k = 0$ by the second identity in (a). Assume (d) holds for some k $(0 \leq k < n)$. If $n-k$ is even, then

$$\Sigma^{(k)} \models d_{n-k-1}(x,x,y) \overset{\text{(c)}}{\approx} d_{n-k}(x,x,y) \overset{\text{(d)}}{\approx} y,$$

so d_{n-k-1} is weakly independent of its first 2 places relative to $\Sigma^{(k)}$. Thus

$$\Sigma^{(k+1)} \models d_{n-k-1}(x,y,z) \approx d_{n-k-1}(z,z,z) \approx z,$$

showing that (d) holds for $k+1$. If $n-k$ is odd, then

$$\Sigma^{(k)} \models d_{n-k-1}(x,y,y) \overset{\text{(b)}}{\approx} d_{n-k}(x,y,y) \overset{\text{(d)}}{\approx} y$$

and

$$\Sigma^{(k)} \models d_{n-k-1}(x,y,x) \overset{\text{(b)}}{\approx} d_{n-k}(x,y,x) \overset{\text{(d)}}{\approx} x,$$

so d_{n-k-1} is weakly independent of its first 2 places relative to $\Sigma^{(k)}$. Thus

$$\Sigma^{(k+1)} \models d_{n-k-1}(x,y,z) \approx d_{n-k-1}(z,z,z) \approx z,$$

and again (d) holds for $k+1$. So, whether k is even or odd, we derive the next instance of (d): $\Sigma^{(k+1)} \models d_{n-(k+1)}(x,y,z) \approx z$.

5. The linear case is decidable

The title of the paper suggests that the derivative test is an easy test for congruence modularity. How easy is it?

George McNulty proves in [23] that the problem of determining if a finite set of identities Σ axiomatizes a congruence modular variety is undecidable. In fact, the results of his paper imply that each of the following problems is undecidable.

(1) Determining if a finite set of idempotent identities Σ axiomatizes a congruence modular variety.
(2) Determining if some function symbol F is weakly independent (or independent) of one of its places relative to Σ.

(3) Determining if Σ' is inconsistent.
(4) Determining if a finitely based variety (given by its basis) realizes a given Σ. (Here Σ can be any fixed finite set of identities which entails an identity of the form $s \approx x$ such that at least two variables x, y occur in the term s.)

On the other hand, one can't help but notice that the derivative test was easy to apply in all the examples given earlier in this paper. We prove in this section that if Σ is a set of identities that is idempotent and linear, then the derivative test is a necessary and sufficient condition for determining if varieties realizing Σ must be congruence modular (Theorem 5.1) or must satisfy a nontrivial congruence identity (Theorem 5.2). We then describe an algorithm for deciding, when Σ is a given finite set of idempotent linear identities, whether the realization of Σ forces congruence modularity or the satisfaction of a nontrivial congruence identity (Corollary 5.3).

We call a term t *linear* if it has at most one occurrence of a function symbol. We call an identity $s \approx t$ *linear* if both s and t are linear. Throughout this section Σ will be a set of idempotent linear identities. Examples of linear identities are those expressing that a function symbol is weakly independent of its i-th place, or fully independent of its i-th place, and the identity $x \approx y$.

If Σ is a consistent set of idempotent linear identities, then a construction in [16] shows that Σ has models of all cardinalities greater than zero. To describe the construction, let V be a set containing an element 0, and let $X = \{x_v \mid v \in V\}$ be a set of variables. For each n-ary function symbol F appearing in Σ and for all $v_1, \ldots, v_n \in V$ define

$$F^{\mathbf{V}}(v_1, \ldots, v_n) = \begin{cases} v & \text{if } \Sigma \models F(x_{v_1}, \ldots, x_{v_n}) \approx x_v, \\ 0 & \text{else.} \end{cases}$$

$\mathbf{V}$ is the algebra whose universe is V and whose basic operations are all operations $F^{\mathbf{V}}$. It is proved in [16] that $\mathbf{V}$ is a model of Σ.

Theorem 5.1. *The following are equivalent for a set Σ of idempotent linear identities.*

(1) *Σ' is inconsistent.*
(2) *Any variety that realizes Σ is congruence modular.*
(3) *The variety axiomatized by Σ is congruence modular.*

Proof. The implication (1)⇒(2) is from Theorem 3.2. The implication (2)⇒(3) is trivial, since the variety axiomatized by Σ realizes Σ. The important part of the proof is (3)⇒(1), which we prove in contrapositive form.

Assume that Σ' is consistent. We will show that the variety axiomatized by Σ is not congruence modular. Let $\mathbf{V}$ and $\mathbf{V}'$ be the models of Σ and Σ' that have the same universe $V = V' = \{0, 1\}$ and are constructed in the manner described before the theorem statement. Since both $\mathbf{V}$ and $\mathbf{V}'$ are models of Σ, the product algebra

$\mathbf{V} \times \mathbf{V}'$ is a model of Σ. We shall show that this product does not have a modular congruence lattice.

For simplicity we will write a pair $(v, w) \in V \times V'$ as vw. Let η and η' be the kernels of the projections of $\mathbf{V} \times \mathbf{V}'$ onto the coordinate algebras $\mathbf{V}$ and $\mathbf{V}'$ respectively. These are the equivalence relations on $V \times V'$ whose associated partitions are $\{\{00, 01\}, \{10, 11\}\}$ and $\{\{00, 10\}, \{01, 11\}\}$ respectively. Let θ denote the equivalence relation on $V \times V'$ whose associated partition is $\{\{00, 01\}, \{10\}, \{11\}\}$. We claim that θ is a congruence of $\mathbf{V} \times \mathbf{V}'$. Since $\theta < \eta$, and both of these equivalence relations are complementary to η', establishing this claim will show that $\mathbf{Con}(\mathbf{V} \times \mathbf{V}')$ is not modular. (Specifically, $\eta \wedge (\eta' \vee \theta) = \eta > \theta = (\eta \wedge \eta') \vee \theta$ is a failure of the modular law.)

If θ is not a congruence, then there must be a function symbol F such that $F^{\mathbf{V}\times\mathbf{V}'}$ is incompatible with θ. After permuting the places of F, we may assume this incompatibility is expressed as

$$F^{\mathbf{V}\times\mathbf{V}'}(\underline{00}, \mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}) \not\equiv F^{\mathbf{V}\times\mathbf{V}'}(\underline{01}, \mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}) \pmod{\theta}. \tag{5.1}$$

Here the bold symbols $\mathbf{xy}$ represent a sequence $(xy, xy, \ldots, xy)$ of pairs of some length (possibly zero) with all pairs equal to xy. The two sides of (5.1) differ only at the single underlined place. Thus, (5.1) expresses that some basic translation does not preserve θ.

Since $\theta \subseteq \eta$, and η is a congruence, the left and right hand sides of (5.1) must be related by η but not by θ. The only two elements so related are 10 and 11, so we must have either

(1) $F^{\mathbf{V}\times\mathbf{V}'}(\underline{00}, \mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}) = 10$ and $F^{\mathbf{V}\times\mathbf{V}'}(\underline{01}, \mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}) = 11$, or
(2) $F^{\mathbf{V}\times\mathbf{V}'}(\underline{00}, \mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}) = 11$ and $F^{\mathbf{V}\times\mathbf{V}'}(\underline{01}, \mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}) = 10$.

By examining first coordinates in $F^{\mathbf{V}\times\mathbf{V}'}(\underline{01}, \mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}) = 10$ or 11 we obtain $F^{\mathbf{V}}(\underline{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}) = 1$, hence

$$\Sigma \models F(x_0, \mathbf{x_0}, \mathbf{x_0}, \mathbf{x_1}, \mathbf{x_1}) \approx \mathbf{x_1}.$$

This shows that $F(p, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t})$ is weakly independent of its first three blocks of variables relative to Σ. Hence $F(p, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t})$ is fully independent of its first three blocks of variables relative to Σ'. But now we discover a contradiction by examining the second coordinates of both expressions from above. In either Case (1) or Case (2) we have

$$F^{\mathbf{V}'}(\underline{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}) \neq F^{\mathbf{V}'}(\underline{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}).$$

This cannot happen in the model $\mathbf{V}'$ of Σ' if, as we have proved, F is independent of its first three blocks of variables relative to Σ'. The contradiction proves that θ is a congruence of $\mathbf{V} \times \mathbf{V}'$, and therefore that $\mathbf{V} \times \mathbf{V}'$ is a model of Σ whose congruence lattice is nonmodular. □

We have an analogue of Theorem 5.1 concerning the satisfaction of a nontrivial congruence identity.

Theorem 5.2. *The following are equivalent for a set Σ of idempotent linear identities.*

(1) *$\Sigma^{(n)}$ is inconsistent for some finite n.*
(2) *Any variety that realizes Σ satisfies a nontrivial congruence identity.*
(3) *The variety axiomatized by Σ satisfies a nontrivial congruence identity.*

Proof. As was the case in the proof of Theorem 5.1, the only nonobvious part of this proof is the part that shows that if $\Sigma^{(n)}$ is consistent for all n, then the variety axiomatized by Σ satisfies no nontrivial congruence identity.

Assume that $\Sigma^{(n)}$ is consistent for all n and let $\Omega = \bigcup_{n<\omega} \Sigma^{(n)}$. Then Ω is a consistent set of idempotent linear identities, and $\Omega' = \Omega$. Let $\mathbf{V}$ be the 2-element model of Ω that is defined before the statement of Theorem 5.1. That is, $\mathbf{V}$ has universe $V = \{0, 1\}$ and for each function symbol F appearing in Ω we have $F^{\mathbf{V}}(v_1, \ldots, v_n) = 1$ if and only if $\Omega \models F(x_{v_1}, \ldots, x_{v_n}) = x_1$.

The fact that $\Omega' = \Omega$ implies that weak independence agrees with independence for all F (relative to Ω). Thus, whenever $\Omega \models F(x_{v_1}, \ldots, x_{v_n}) = x_1$ holds, the variable x_1 appears in every place upon which F depends. This means that $F^{\mathbf{V}}(v_1, \ldots, v_n) = 1$ if and only if $v_i = 1$ for each place i upon which F depends. If $I \subseteq \{1, \ldots, n\}$ is the set of places upon which F depends and $\wedge$ is the meet on $\{0, 1\}$ for the order $0 < 1$, then $F^{\mathbf{V}}$ agrees with $\bigwedge_{i \in I} x_i$ on $\{0, 1\}$. Since $\mathbf{V}$ is a model of Ω, this proves that Ω can be realized by a 2-element semilattice. Hence Ω (and therefore Σ) cannot axiomatize a variety satisfying a nontrivial congruence identity. □

Corollary 5.3. *Both of the following problems are decidable: For a finite set Σ of idempotent linear identities*

(1) *determine if the realization of Σ implies congruence modularity;*
(2) *determine if the realization of Σ implies the satisfaction of a nontrivial congruence identity.*

Proof. We begin by explaining how to decide if $\Sigma \models \varphi$ when $\Sigma \cup \{\varphi\}$ is a finite set of linear identities. (This part does not need the assumption that Σ is idempotent.)

Let X be a set of variables that includes all variables occurring in Σ. The *weak closure of Σ in the variables X* is the smallest set $\overline{\Sigma}$ of linear identities containing Σ for which

(i) $(t \approx t) \in \overline{\Sigma}$ for all linear terms t with variables from X and with function symbol (if any) occurring in Σ;
(ii) if $(s \approx t) \in \overline{\Sigma}$, then $(t \approx s) \in \overline{\Sigma}$;
(iii) if $(r \approx s) \in \overline{\Sigma}$ and $(s \approx t) \in \overline{\Sigma}$, then $(r \approx t) \in \overline{\Sigma}$;

(iv) if $(s \approx t) \in \overline{\Sigma}$ and $\gamma\colon X \to X$ is a function, then $(s[\gamma] \approx t[\gamma]) \in \overline{\Sigma}$, where $s[\gamma]$ denotes the linear term obtained from s by replacing each variable $x \in X$ with $\gamma(x) \in X$.

Write $\Sigma \vdash_X \varphi$ if φ belongs to the weak closure of Σ in the variables X. We say that X is *large enough* for Σ if

(a) X contains all variables occurring in Σ,
(b) $|X| \geq 2$, and
(c) $|X| \geq \mathrm{arity}(F)$ for any function symbol F occurring in Σ.

We will assume throughout that x, y denote distinct variables in X.

David Kelly's Completeness Theorem [19, 16] states[2] that if X is large enough for $\Sigma \cup \{\varphi\}$, then $\Sigma \models \varphi$ if and only if $\Sigma \vdash_X x \approx y$ or $\Sigma \vdash_X \varphi$. Thus, in order to decide if $\Sigma \models \varphi$ when $\Sigma \cup \{\varphi\}$ is linear we must test whether $x \approx y$ or φ belong to the weak closure $\overline{\Sigma}$ of Σ in the variables X. By definition, $\overline{\Sigma}$ is an equivalence relation on the finite set $\mathcal{T}_X$ of linear terms whose function symbols are from Σ and whose variables are from X; in fact, $\overline{\Sigma}$ is the least equivalence relation on $\mathcal{T}_X$ containing Σ that is closed under all variable replacements $X \to X$. Hence $\overline{\Sigma}$ can be computed from Σ, which allows us to decide $\Sigma \models \varphi$.

Now we describe decision procedures for items (1) and (2) of the corollary. Given a finite set Σ of idempotent linear identities, let X be a finite set of variables that is large enough for Σ and satisfies $|X| \geq 1 + \mathrm{arity}(F)$ for all function symbols F occurring in Σ. Then X contains sufficiently many variables to write down identities expressing that some F is independent of one of its places. Therefore X is large enough for Σ', and hence for all $\Sigma^{(n)}$.

We saw earlier in this proof that the weak closure $\overline{\Sigma}$ of Σ in the variables X is an equivalence relation on the finite set $\mathcal{T}_X$, which can be computed from Σ. Therefore, by inspecting the equivalence class of $x \in X$ one can (i) determine if $\Sigma \vdash_X x \approx y$ (i.e., if Σ is inconsistent), (ii) find all instances of weak independence relative to Σ, and hence (iii) compute Σ'. Repeating the same procedure for Σ' one can determine if Σ' is inconsistent. By Theorem 5.1 this proves that problem (1) is decidable.

Since $\Sigma \subseteq \Sigma' \subseteq \cdots \subseteq \Sigma^{(i)} \subseteq \Sigma^{(i+1)} \subseteq \cdots$ and X is sufficiently large for each $\Sigma^{(i)}$ $(i \geq 0)$, we get that their weak closures in the variables X form an ascending chain

$$\overline{\Sigma} \subseteq \overline{\Sigma'} \subseteq \cdots \subseteq \overline{\Sigma^{(i)}} \subseteq \overline{\Sigma^{(i+1)}} \subseteq \cdots \tag{5.2}$$

of equivalence relations on the finite set $\mathcal{T}_X$. By construction, if $\overline{\Sigma^{(i)}} = \overline{\Sigma^{(i+1)}}$ for some i, then $\overline{\Sigma^{(i)}} = \overline{\Sigma^{(k)}}$ for all $k \geq i$. Therefore the chain (5.2) stabilizes in at most $|\mathcal{T}_X|$ steps. Let $\overline{\Sigma^{(n)}}$ denote the largest member of the chain. Our earlier discussion shows that $\overline{\Sigma}, \overline{\Sigma'}, \ldots, \overline{\Sigma^{(n)}}$ are computable from Σ. It follows from Theorem 5.2 that

[2]In fact, Kelly proves his completeness theorem for *basic identities*, which are more general than linear identities.

the realization of Σ implies the satisfaction of a nontrivial congruence identity if and only if $\overline{\Sigma^{(n)}}$ is the total relation on $\mathcal{T}_X$. This proves that problem (2) is decidable. □

In [3], Gábor Czédli and Ralph Freese describe an algorithm to determine if a lattice identity ε implies modularity as a congruence identity. The algorithm has two main steps. In the first step one replaces all joins in ε with 4-fold compositions to obtain a relational identity ε^* in the symbols $\{\circ, \cap\}$; then one finds a strong Maltsev condition Σ equivalent to the satisfaction of ε^* as a congruence condition. In the second step, one tests if Σ can be realized in Polin's variety. The identity ε implies modularity as a congruence identity if and only if Σ cannot be so realized. (This second step is decidable because Σ is a finite set of identities and Polin's variety is locally finite.) The second step can be replaced with the derivative test, since Σ is idempotent and linear. In fact, the derivative test may be viewed as a generalization of the Czédli–Freese result from the setting of congruence identities implying modularity to the setting of idempotent, linear, strong Maltsev conditions implying modularity.

Problem 5.4. Find derivative-like tests for other Maltsev conditions, such as congruence meet or join semidistributivity, or congruence n-permutability for some n.

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