On Maltsev Conditions associated with omitting certain types of local structures

Abstract

Tame congruence theory identifies six Maltsev conditions associated with locally finite varieties omitting certain subsets of types of local behaviour. Extending a result of Siggers, we show that of these six Maltsev conditions only two of them are equivalent to strong Maltsev conditions for locally finite varieties. Besides omitting the unary type [18], the only other of these conditions that is strong is that of omitting the unary and affine types.

We also present a Maltsev condition that is similar in form to having a Taylor term but that is associated with omitting the unary, affine, and semilattice local types.

1. Introduction

A powerful tool for classifying varieties of algebras emerged in the 1960s, motivated by results of Maltsev [15] on congruence permutable varieties and by Jónsson [11] on congruence distributive varieties. They show that these congruence conditions on varieties can be expressed via what is now called a Maltsev condition. Since then, many important properties of varieties have been shown to be equivalent to Maltsev conditions.

Definition 1.1 ([9, 12]) *1. Let $\mathcal{U}$ and $\mathcal{V}$ be varieties and suppose that the operation symbols of $\mathcal{U}$ are $\{f_i : i \in I\}$. We say that $\mathcal{U}$ is interpretable in $\mathcal{V}$, and write $\mathcal{U} \leq \mathcal{V}$, if for every $i \in I$ there is a $\mathcal{V}$-term t_i of the same arity as f_i such that for all $\mathbf{A} \in \mathcal{V}$, the algebra $\langle A, t_i^{\mathbf{A}}(i \in I)\rangle$ is a member of $\mathcal{U}$.*

2. *If $\mathcal{U}$ is a finitely presented variety, i.e., it has finitely many operation symbols and is finitely axiomatized, then the class of all varieties $\mathcal{V}$ with $\mathcal{U} \leq \mathcal{V}$ is called the strong Maltsev class defined by $\mathcal{U}$, and the condition $\mathcal{U} \leq \mathcal{V}$ on $\mathcal{V}$ is called the strong Maltsev condition defined by $\mathcal{U}$.*

3. *If $\mathcal{U}_i$, $i \geq 0$ is a decreasing sequence of finitely presented varieties, relative to interpretability, then the class $\{\mathcal{V} : \mathcal{U}_i \leq \mathcal{V}$ for some $i\}$ is called the Maltsev class defined by this sequence, and the associated condition on varieties is called the Maltsev condition defined by this sequence.*

4. *An operation $f(\vec{x})$ on a set A is idempotent if the equation $f(x,x,\dots,x) = x$ holds. A term $t(\vec{x})$ of an algebra or variety is idempotent if the associated operation is, and we call an algebra or variety idempotent if all of its terms are.*

5. *We say that a Maltsev condition is proper if it is not equivalent to a strong Maltsev condition. A (strong) Maltsev condition is idempotent if the (variety) varieties used to define it (is) are.*

In [10] a theory of the local structure of finite algebras is developed. Hobby and McKenzie show that relative to a pair of congruences $\alpha \prec \beta$ of a finite algebra $\mathbf{A}$, the local behaviour of the non-trivial β classes, modulo α, falls into one of five possible types:

1. Unary
2. Affine
3. 2-element Boolean algebra
4. 2-element Lattice
5. 2-element Semilattice

and that this local behaviour is uniform across all β classes.

This allows one to speak of the type of the pair (α, β) as being one of the five types listed above. For example to assert that the type of (α, β) is $\mathbf{3}$ means that locally, the β classes behave as 2-element Boolean algebras, modulo α. Extending this further, we can define the type set of a finite algebra and the type set of a variety or of any collection of algebras. We may also speak of an algebra or class of algebras admitting or omitting certain types.

Definition 1.2 *Let $\mathbf{A}$ be a finite algebra and $\mathcal{K}$ a class of algebras. The type set of $\mathbf{A}$, denoted typ$\{\mathbf{A}\}$ is defined to be the set of all types $\mathbf{i}$ such that the type of (α, β) is $\mathbf{i}$ for some pair of congruences $\alpha \prec \beta$ of $\mathbf{A}$. The type set of $\mathcal{K}$, denoted typ$\{\mathcal{K}\}$ is defined to be the union of typ$\{\mathbf{A}\}$ for all finite $\mathbf{A} \in \mathcal{K}$.*

In Chapter 9 of [10] six type-omitting conditions are studied and, remarkably, are shown to be equivalent to idempotent Maltsev conditions, for locally finite varieties. They are also shown to have natural expressions in terms of familiar conditions on the congruence lattices of the members of the varieties. We note that in [10] none of the presentations of the Maltsev conditions for these six classes are strong. The six conditions are:

Name	Type Omitting Condition	Other Properties
$\mathcal{M}_1$	$\{\mathbf{1}\}$	largest non-trivial idempotent Maltsev class
$\mathcal{M}_2$	$\{\mathbf{1}, \mathbf{5}\}$	equivalent to satisfying a non-trivial congruence identity (see [12])
$\mathcal{M}_3$	$\{\mathbf{1}, \mathbf{4}, \mathbf{5}\}$	n-permutable varieties
$\mathcal{M}_4$	$\{\mathbf{1}, \mathbf{2}\}$	congruence meet semi-distributive varieties
$\mathcal{M}_5$	$\{\mathbf{1}, \mathbf{2}, \mathbf{5}\}$	congruence join semi-distributive varieties (see [12])
$\mathcal{M}_6$	$\{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}\}$	n-permutable and congruence join semi-distributive varieties

Some of these conditions have particularly nice descriptions in terms of interpretability and term conditions. The following definition and theorem expand on this.

Definition 1.3 *1. The term t is a Taylor term for* $\mathbf{A}$ *or* $\mathcal{V}$ *if it is idempotent and for each* $1 \leq i \leq n$*, an equation in the variables* $\{x, y\}$ *of the form* $t(a_1, \dots, a_n) = t(b_1, \dots, b_n)$ *holds, where* $a_i \neq b_i$.

2. *The term t is a Hobby-McKenzie term for* $\mathbf{A}$ *or* $\mathcal{V}$ *if it is idempotent and for each* $U \subseteq \{1, \dots, n\}$*, an equation in the variables* $\{x, y\}$ *of the form* $t(a_1, \dots, a_n) = t(b_1, \dots, b_n)$ *holds, where* $\{a_i : i \in U\} \neq \{b_i : i \in U\}$.

3. *The term t is called a weak near unanimity term for* $\mathbf{A}$ *or* $\mathcal{V}$ *if it is idempotent and the equations*

$$t(y, x, \dots, x) = t(x, y, x, \dots, x) = \cdots = t(x, x, \dots, x, y)$$

hold.

Theorem 1.4 *Let* $\mathcal{V}$ *be a locally finite variety.*

1. $\mathcal{V} \in \mathcal{M}_1$ *if and only if for some* $n > 1$, $\mathcal{V}$ *has an n-ary Taylor term if and only if for some* $n > 1$, $\mathcal{V}$ *has an n-ary weak near unanimity term.*

2. $\mathcal{V} \in \mathcal{M}_2$ *if and only if for some* $n > 1$, $\mathcal{V}$ *has an n-ary Hobby-McKenzie term.*

3. $\mathcal{V} \in \mathcal{M}_4$ *if and only if for all* $n > 2$, $\mathcal{V}$ *has an n-ary weak near unanimity term.*

PROOF: The proof of the first claim follows from Lemma 9.4 and Theorem 9.6 of [10] and Theorem 1.1 of [16]. The proof of the second follows from Lemma 9.5 and Theorem 9.8 of [10] and the proof of the last claim follows from Theorem 1.2 of [16] and the main result of [1] or [5]. •

A surprising result of Siggers [18], announced in 2008, and based on earlier work with Nešetřil [17], is that the class $\mathcal{M}_1$ can in fact be defined by a strong Maltsev condition.

Theorem 1.5 ([18]) *Let* $\mathcal{V}$ *be a locally finite variety. Then* $\mathcal{V}$ *omits the unary type if and only if it has a 6-ary idempotent term t such that* $\mathcal{V}$ *satisfies the equations*

$$t(x, x, x, x, y, y) = t(x, y, x, y, x, x) \quad \text{and} \quad t(y, y, x, x, x, x) = t(x, x, y, x, y, x).$$

One direction of this theorem follows by noting that any term t that satisfies the stated conditions is a Taylor term. In the next section we will present a proof of a variant of this theorem that uses a deep result of Barto and the second author. We will also establish a similar result for the class $\mathcal{M}_4$.

2. Strong Maltsev Conditions

Shortly after the announcement of Siggers's result it was noted by Marković and McKenzie that one could replace the 6-ary term of Siggers by one of several types of 4-ary terms. Their proof employs a deep result of Barto, Niven, and the second author [4] on the complexity of the graph homomorphism problem. We make use of a theorem of Barto and the second author on cyclic terms to establish one version of this result.

A term $t(x_1, \ldots, x_n)$ of an algebra or variety is cyclic if it is idempotent and satisfies the equation $t(x_1, x_2, \ldots, x_{n-1}, x_n) = t(x_2, x_3, \ldots, x_n, x_1)$. Note that cyclic terms are special examples of weak near unanimity and Taylor terms.

Theorem 2.1 ([2]) *Let* $\mathbf{A}$ *be a finite algebra. Then* $\mathsf{V}(\mathbf{A})$ *omits the unary type if and only if for all prime numbers* $p > |A|$, $\mathbf{A}$ *has a* p*-ary cyclic term operation.*

Corollary 2.2 *A locally finite variety* $\mathcal{V}$ *omits the unary type if and only if it has a 4-ary idempotent term operation* t *that satisfies the identities:*

$$t(y, y, x, x) = t(x, y, y, x) = t(x, x, x, y).$$

PROOF: It suffices to show that the free algebra $\mathbf{F}$ in $\mathcal{V}$ on 2 generators has such a term. Let p be some prime number larger than $|F|$ of the form $5k + 3$ for some k and let $c(x_1, \ldots, x_p)$ be a cyclic term of $\mathbf{F}$. Define $t(x, y, z, w)$ to be the term

$$c(x, x, \ldots, x, y, y, \ldots, y, z, z, \ldots z, w, w, \ldots w),$$

where the variables x and z occur $k+1$ times, y occurs k times and w occurs $2k+1$ times. Using that c is cyclic, it is easy to verify that $t(x, y, z, w)$ satisfies the stated equations in $\mathbf{F}$ and hence in $\mathcal{V}$.

Conversely, any term that satisfies the stated equations is a Taylor term and so any locally finite variety having such a term operation omits the unary type. •

In order to prove that the class $\mathcal{M}_4$ can be defined by a strong Maltsev condition we must first take a detour in to ideas and results on the constraint satisfaction problem (CSP). For some background on the CSP, the reader is encouraged to consult [6] and more generally [8].

We present some standard definitions related to the CSP that have been suitably modified to meet our needs in this paper.

Definition 2.3 *Let* $\mathbf{A}$ *be a finite algebra. An instance of the constraint satisfaction problem over* $\mathbf{A}$ *is a triple* $P = (V, A, \mathcal{C})$ *where*

- V *is a non-empty, finite set of variables and*
- $\mathcal{C}$ *is a set of constraints* $\{C_1, \ldots, C_q\}$ *where each* C_i *is a pair* (S_i, R_i) *with*
  - S_i *a non-empty subset of* V *called the scope of* C_i*, and*
  - R_i *is a subuniverse of the algebra* $\mathbf{A}^{S_i}$*, called the constraint relation of* C_i*.*

We denote by $\mathsf{CSP}(\mathbf{A})$ *the class of all instances of the* CSP *over* $\mathbf{A}$.

A solution of P *is a member* $\vec{s}$ *of* A^V *such that for all* $i \leq q$, $\mathrm{proj}_{S_i}(\vec{s})$, *the projection of* $\vec{s}$ *onto the coordinates* S_i, *is a member of* R_i.

Definition 2.4 *Let* $\mathbf{A}$ *be a finite algebra and* $P = (V, A, \mathcal{C}) \in \mathsf{CSP}(\mathbf{A})$.

1. *For* $k > 0$, P *is* k*-minimal if*
   - *For each subset* I *of* V *of size at most* k, *there is some constraint* (S, R) *in* $\mathcal{C}$ *such that* $I \subseteq S$ *and*
   - *If* (S_1, R_1) *and* (S_2, R_2) *are constraints in* $\mathcal{C}$ *and* $I \subseteq S_1 \cap S_2$ *has size at most* k *then* $\mathrm{proj}_I(R_1) = \mathrm{proj}_I(R_2)$.

   For $I \subseteq V$ *with* $|I| \leq k$, *these conditions allow us to define* P_I *to be the projection of* R *onto* I *for some (or any)* $(S, R) \in \mathcal{C}$ *with* $I \subseteq S$.
2. P *is* $(2,3)$*-minimal if it is 2-minimal and for every 3 element subset* $\{i, j, k\}$ *of* V *and every pair* $(a, b) \in P_{\{i,j\}}$ *there is some* $c \in A$ *such that* $(a, c) \in P_{\{i,k\}}$ *and* $(b, c) \in P_{\{j,k\}}$.
3. $\mathbf{A}$ *is said to be of relational width* k *(or width* $(2,3)$*) if every* k*-minimal* (($2,3$)*-minimal) instance of* $\mathsf{CSP}(\mathbf{A})$ *whose constraint relations are all non-empty has a solution.*

We invoke two key results from the theory of the CSP in order to proof the main result of this section.

Theorem 2.5 ([3, 5]) *Let* $\mathbf{A}$ *be a finite idempotent algebra.*

1. *[3, 5]* $\mathbf{A}$ *is of relational width* k *for some* $k > 1$ *if and only if it is of relational width 3 if and only if* $\mathsf{V}(\mathbf{A})$ *omits the unary and affine types.*
2. *[3]* $\mathbf{A}$ *is of width* $(2,3)$ *if and only if* $\mathsf{V}(\mathbf{A})$ *omits the unary and affine types.*

Lemma 2.6 ([3]) *A finite algebra* $\mathbf{A}$ *generates a variety that omits the unary and affine types if and only if for some* $k > 0$ *it has weak near unanimity terms* v *and* w *of arities* k *and* $k+1$ *respectively and such that the equation* $v(y, x, \ldots, x) = w(y, x, \ldots, x)$ *holds in* $\mathbf{A}$.

Theorem 2.7 *A locally finite variety omits the unary and affine types if and only if it has 3-ary and 4-ary weak near unanimity terms* $v(x, y, z)$ *and* $w(x, y, z, w)$ *that satisfy the equation* $v(y, x, x) = w(y, x, x, x)$.

PROOF: It follows from the previous Lemma that any locally finite variety with terms as described in the statement of the theorem generates a variety that omits the unary and affine types. Conversely, if $\mathcal{V}$ omits the unary and affine types then we will build a $(2,3)$-minimal instance of the CSP over some finite algebra in $\mathcal{V}$ that, by Theorem 2.5, is guaranteed to have a solution. We may assume that $\mathcal{V}$ is idempotent since omitting the unary and affine types is determined by the idempotent terms of the variety.

Our construction is a variation of one found in [13] to show that finite algebras of relational width k must have k-ary weak near unanimity terms. Let $\mathbf{F}$ be the $\mathcal{V}$-free algebra generated by $\{x, y\}$, let R be the subuniverse of $\mathbf{F}^3$ generated by

$$\{(y, x, x), (x, y, x), (x, x, y)\},$$

and S the subuniverse of $\mathbf{F}^4$ generated by

$$\{(y, x, x, x), (x, y, x, x), (x, x, y, x), (x, x, x, y)\}.$$

Let $n > 4|F|$ and let $P = (V, F, \mathcal{C})$ be the following instance of the CSP:

- $V = \{x_1, \ldots, x_n\}$,
- $\mathcal{C}$ consists of constraints C_I for all $I \subseteq V$ with $|I| = 3$ or 4, where $C_I = (I, R)$ if $|I| = 3$ and (I, S) if $|I| = 4$.

Since the subuniverses R and S are totally symmetric, and their projections onto any pair of coordinates are the same, it follows that P is a $(2, 3)$-minimal instance over $\mathbf{F}$. Since $\mathbf{F}$ generates a variety that omits the unary and affine types then, by Theorem 2.5, we conclude that P has a solution $\vec{s} \in F^V$. Since $n > 4|F|$ then by the Pigeon-Hole Principle it follows that $\vec{s}$ is constant on some $I \subseteq V$ with $|I| = 4$, say over the coordinates in I, $\vec{s}$ takes on the value $x \circ y \in F$.

It follows that, since $\vec{s}$ is a solution of P and P contains the constraints (J, R) and (I, S) (where J is any 3 element subset of I), $(x \circ y, x \circ y, x \circ y) \in R$ and $(x \circ y, x \circ y, x \circ y, x \circ y) \in S$

Since R is generated by

$$\{(y, x, x), (x, y, x), (x, x, y)\},$$

and S by

$$\{(y, x, x, x), (x, y, x, x), (x, x, y, x), (x, x, x, y)\}$$

we conclude that there are terms $v(x, y, z)$ and $w(x, y, z, u)$ of $\mathcal{V}$ such that the following equations hold in $\mathbf{F}$:

$$v(y, x, x) = v(x, y, x) = v(x, x, y) = x \circ y$$

and

$$w(y, x, x, x) = w(x, y, x, x) = w(x, x, y, x) = w(x, x, x, y) = x \circ y.$$

Thus v and w are the desired terms of $\mathcal{V}$. •

If one would rather deal with k-minimality instead of $(2, 3)$-minimality, then the above proof can easily be modified to show that algebras of relational width 3 must have 4-ary and 5-ary weak near unanimity terms r and s with $r(y, x, x, x) = s(y, x, x, x, x)$. This, of course, provides another strong Maltsev condition for omitting the unary and affine types.

Corollary 2.8 *The class $\mathcal{M}_4$ is defined by an idempotent strong Maltsev condition.*

Proof: Let $\mathcal{U}$ be the finitely presented variety with a 3-ary operation symbol v and a 4-ary operation symbol w defined by the equations that assert that v and w are weak near unanimity terms and that $v(y, x, x) = w(y, x, x, x)$. By the previous Theorem we have that a locally finite variety $\mathcal{V}$ is in $\mathcal{M}_4$ if and only if $\mathcal{U} \leq \mathcal{V}$. •

3. Proper Maltsev Conditions

In this section we present a construction that we use to show that none of the other classes $\mathcal{M}_i$, $i \neq 1, 4$ can be defined by a strong Maltsev condition. Our construction is based on an example found in Section 3 of [7].

Definition 3.1 *Let $n > 0$ and $1 \leq i \leq n$.*

1. *Let $\mathbf{A}[i,n]$ be the algebra with universe $\{0,1\}$ and whose only basic operation is the $2n+1$-ary operation $t_{(i,n)}$ defined by:*

$$t_{(i,n)}(x_0, x_1, \ldots, x_{2n-1}, x_{2n}) = x_0 \wedge (x_1 \wedge x_2) \wedge \cdots \wedge (x_{2i-3} \wedge x_{2i-2}) \wedge (\overline{x_{2i-1}} \vee x_{2i}).$$

2. *Let $\mathbf{A}_n$ be the cartesian product $\prod_{i=1}^{n} \mathbf{A}[i,n]$ and let $\mathcal{V}_n$ be the variety generated by $\mathbf{A}_n$. Denote the sole operation symbol of $\mathcal{V}_n$ by t_n.*

We observe that each $\mathbf{A}[i,n]$ is term equivalent to the algebra $(\{0,1\}, x \wedge (\bar{y} \vee z))$ and so generates a variety that is congruence distributive and 3-permutable. The set of term operations of this algebra is equal to the set of all idempotent operations $f(x_1, \ldots, x_m)$ on $\{0,1\}$ such that for some $j \leq m$, $f(\vec{x}) \leq x_j$ for all $\vec{x} \in \{0,1\}^m$.

We first show that for any $n > 0$, the typeset of $\mathcal{V}_n$ is $\{\mathbf{3}\}$, i.e., the only type admitted by a finite algebra in $\mathcal{V}_n$ is the Boolean type. Note that the term $t_n(x, x, y, y, \ldots, y)$ interprets in $\mathbf{A}_n$ as $x \wedge y$, the standard meet operation, on $\{0,1\}^n$.

Theorem 3.2 *For any $n > 0$, the typeset of $\mathcal{V}_n$ is $\{\mathbf{3}\}$.*

Proof: From Theorem 9.15 of [10] we know that $\mathcal{V}_n$ has typeset $\{\mathbf{3}\}$ if and only if for some $k > 0$ it has a sequence of 4-ary terms $f_i(x, y, z, w)$, $0 \leq i \leq k$ that satisfy the equations:

$$\begin{aligned} x &= f_0(x, y, y, z), \\ f_i(x, x, y, x) &= f_{i+1}(x, y, y, x), \text{ for each } i < k, \\ f_i(x, x, y, y) &= f_{i+1}(x, y, y, y), \text{ for each } i < k, \\ f_k(x, x, y, z) &= z. \end{aligned}$$

We claim that the following sequence of terms of $\mathcal{V}_n$ satisfies these equations, for $k = 2n$. For $0 \leq i < n$, let

$$f_i(x, y, z, w) = t_n(x, x, x, \ldots, x, x, y, z, w, w, \ldots, w, w),$$

where w occurs $2i$ times, let

$$f_n(x, y, z, w) = t_n(x \wedge (y \wedge w), z, y, w, w, \ldots, w)$$

and for $0 < i \leq n$, let

$$f_{n+i}(x, y, z, w) = f_{n-i}(w, y, x, z).$$

One can verify that these terms do indeed satisfy the above equations by showing that for each $i \leq n$ they hold in the algebra $\mathbf{A}[i,n]$. We leave this as an exercise for the reader. •

Definition 3.3 *Let $n > 0$ and $t(x_1, \ldots, x_m)$ be a term of $\mathcal{V}_n$. For $1 \le i \le n$ and $1 \le j \le m$, we define the variable x_j of t to be of sort i if the term t depends on the variable x_j in the algebra $\mathbf{A}[i, n]$.*

Lemma 3.4 *Let $n > 0$ and $t(x_1, \ldots, x_m)$ be a term of $\mathcal{V}_n$. If $m < n$ then there is some $i \le n$ such that in $\mathbf{A}[i, n]$, the interpretation of $t(x_1, \ldots, x_m)$ is equal to the meet of some of the variables in $\{x_1, \ldots, x_m\}$.*

PROOF: For $1 \le i \le n$, let $S_i \subseteq \{x_1, \ldots, x_m\}$ be the set of variables of t of sort i. We first establish the following Claim.

Claim: Let $1 \le i < n$.

1. The interpretation of the term $t(\vec{x})$ in the algebra $\mathbf{A}[i+1, n]$ is equal to an operation of the form $p(\vec{x}) \wedge q(\vec{x})$, where $p(\vec{x}) = \bigwedge_{x \in S_i} x$ and q is some idempotent operation on $\{0, 1\}$.
2. $S_i \subseteq S_{i+1}$.

We prove this claim by induction on the definition of the term t. For t a projection, both parts of the claim are clearly true. Suppose that t can be written as

$$t_n(r_0(\vec{x}), r_1(\vec{x}), r_2(\vec{x}), \ldots, r_{2n-1}(\vec{x}), r_{2n}(\vec{x}))$$

for some terms r_k and that the claim holds for these terms. Since $t_n(x_0, x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ doesn't depend in $\mathbf{A}[i, n]$ on the variables x_k for $k > 2i$, then S_i is contained in the union of the sets of variables of sort i of the terms r_l for $0 \le l \le 2i$. By induction, in $\mathbf{A}[i+1, n]$, the terms $r_l(\vec{x})$, for $0 \le l \le 2i$, can be written as $p_l(\vec{x}) \wedge q_l(\vec{x})$ where $p_l(\vec{x})$ is the meet of the variables of r_l of sort i and $q_l(\vec{x})$ is some idempotent operation on $\{0, 1\}$.

By appealing to the definition of $t_{(i+1,n)}$, the interpretation of t_n in $\mathbf{A}[i+1, n]$, it follows that in $\mathbf{A}[i+1, n]$, $t(\vec{x})$ can be written as

$$\left[\bigwedge_{0 \le l \le 2i} p_l(\vec{x})\right] \wedge \left(\left[\bigwedge_{0 \le l \le 2i} q_l(\vec{x})\right] \wedge (\overline{r_{2i+1}(\vec{x})} \vee r_{2i+2}(\vec{x}))\right).$$

From this it follows that in $\mathbf{A}[i+1, n]$ the interpretation of $t(\vec{x})$ can be written as claimed since all of the elements of S_i appear in the meet of the $p_l(\vec{x})$, $0 \le l \le 2i$.

Since any variable x of t of sort i appears as a meet in the representation of t in $\mathbf{A}[i+1, n]$ it follows that in this algebra, t depends on x. Thus x is also of sort $i+1$ for t and so $S_i \subseteq S_{i+1}$.

Since t is idempotent, then $S_1 \ne \emptyset$ and since $m < n$ then there must be some i with $1 \le i < n$ such that $S_i = S_{i+1}$. From the first part of the Claim it follows that in $\mathbf{A}[i+1, n]$ the interpretation of $t(\vec{x})$ can be written in the form $(\bigwedge_{x \in S_i} x) \wedge q(\vec{x})$ for some idempotent operation q on $\{0, 1\}$. Since $S_i = S_{i+1}$ then in fact in $\mathbf{A}[i+1, n]$ we have that $t(\vec{x})$ equals $(\bigwedge_{x \in S_{i+1}} x) \wedge q(\vec{x})$. It is now straightforward to verify that indeed $t(\vec{x}) = \bigwedge_{x \in S_{i+1}} x$ in $\mathbf{A}[i+1, n]$. •

Theorem 3.5 *Any strong Maltsev condition satisfied by $\mathcal{V}_n$ for all $n > 0$ is also satisfied by the variety of meet semilattices.*

Proof: Since the variety of meet semilattices and $\mathcal{V}_n$ are idempotent for all $n > 0$ then we need only consider idempotent strong Maltsev conditions $\mathcal{U}$ in this proof. By making use of the idempotency of $\mathcal{U}$ we may assume that it can be presented in the form $\langle h(x_1, \ldots, x_m), \Sigma \rangle$ for some $m > 0$ and some finite set of equations Σ in the operation h. For example, if some idempotent finitely presented variety has operations $s(x_1, x_2, x_3)$ and $t(x_1, x_2)$ then an equivalent variety (with respect to interpretability) may be obtained by replacing s and t by a 6-ary operation $r(x_1, \ldots, x_6)$ and replacing all occurrences of $s(x_1, x_2, x_3)$ and $t(x_1, x_2)$ in the equations defining $\mathcal{V}$ by $r(x_1, x_1, x_2, x_2, x_3, x_3)$ and $r(x_1, x_1, x_1, x_2, x_2, x_2)$ respectively. Further details of this reduction can be found in the proof of Lemma 9.4 of [10] and also in [12].

To prove the theorem, it will suffice to show that if $\mathcal{U} \leq \mathcal{V}_n$ for some $n > m$ then $\mathcal{U} \leq$ *Semilattices*, the variety of meet semilattices. This follows from the previous lemma, since if $n > m$ and the $\mathcal{V}_n$-term $t(x_1, \ldots, x_m)$ gives rise to an interpretation of $\mathcal{U}$ in $\mathcal{V}_n$ then for some $i \leq n$ the interpretation of t in $\mathbf{A}[i, n]$ is the meet of some of the variables in $\{x_1, \ldots, x_m\}$. Thus the 2-element meet semilattice has a term that also satisfies the equations in Σ and so the variety of meet semilattices interprets the variety $\mathcal{U}$. •

Corollary 3.6 *Of the classes of locally finite varieties $\mathcal{M}_i$, $1 \leq i \leq 6$, only $\mathcal{M}_1$ and $\mathcal{M}_4$ can be defined by strong Maltsev conditions.*

Proof: That $\mathcal{M}_1$ and $\mathcal{M}_4$ can be defined by strong Maltsev conditions is proved in Corollaries 2.2 and 2.8. Note that if $1 \leq i \leq 6$ and $i \neq 1, 4$ then *Semilattices* $\notin \mathcal{M}_i$. If $\mathcal{U}$ is a strong Maltsev condition that is satisfied by all of the varieties in $\mathcal{M}_i$ then by Theorem 3.2 it is satisfied by $\mathcal{V}_n$ for all $n > 0$. Then by Theorem 3.5, the variety of semilattices also satisfies $\mathcal{U}$ and hence the strong Maltsev condition $\mathcal{U}$ cannot define the class $\mathcal{M}_i$. •

4. A Maltsev condition that implies omitting types 1, 2, and 5

Theorem 1.4 states that a locally finite variety omits the unary and meet semilattice types if and only if it has a Hobby-McKenzie term. In fact, Lemma 9.5 of [10] shows that any variety has a Hobby-McKenzie term if and only if it satisfies some idempotent Maltsev condition that fails in the variety of meet semilattices. In this section we present a variation on the definition of a Hobby-McKenzie term that we will show is related to omitting the unary, affine, and semilattice types. But first, here is a characterization of locally finite varieties that omit these types.

A variety is congruence join semidistributive (or satisfies $\mathsf{SD}(\vee)$) if for all $\mathbf{A} \in \mathcal{V}$ and all α, β, $\gamma \in \mathrm{Con}\,(\mathbf{A})$, if $\alpha \vee \beta = \alpha \vee \gamma$ then $\alpha \vee \beta = \alpha \vee (\beta \wedge \gamma)$.

Theorem 4.1 ([10, 12]) *Let $\mathcal{V}$ be a variety. The following are equivalent:*

1. *$\mathcal{V}$ is congruence join semidistributive.*
2. *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in any nontrivial variety of modules and in the variety of meet semilattices.*

3. *For some $n > 0$, $\mathcal{V}$ has terms $p_i(x, y, z)$ for $0 \le i \le n$ which satisfy the identities:*

$$\begin{aligned} p_0(x,y,z) &= x \\ p_n(x,y,z) &= z \\ p_i(x,y,y) &= p_{i+1}(x,y,y) \text{ and } p_i(x,y,x) = p_{i+1}(x,y,x) \text{ for all } i \text{ even} \\ p_i(x,x,y) &= p_{i+1}(x,x,y) \text{ for all } i \text{ odd} \end{aligned}$$

If $\mathcal{V}$ is locally finite, then these conditions are equivalent to $\mathcal{V}$ omitting the unary, affine, and semilattice types.

Definition 4.2 *An idempotent term $t(x_0, \dots, x_{n-1})$ of a variety $\mathcal{V}$ is called an $\mathsf{SD}(\vee)$-term iff for all subsets $S \subseteq \{x_1, \dots, x_n\}$,*

> *$\mathcal{V}$ satisfies an equation in the variables $\{x, y\}$ of the form $t(a_1, \dots, a_n) = t(b_1, \dots, b_n)$ where $\{a_i : i \in S\} = \{x\}$, $\{b_i : i \in S\} = \{x, y\}$ and there is a unique $i \in S$ with $b_i = y$.*

For future reference, we will denote the above condition on the subset S by $(*)_S$.

Proposition 4.3 *The variety of meet semilattices and any non-trivial variety of modules do not have $\mathsf{SD}(\vee)$-terms.*

Proof: Since an $\mathsf{SD}(\vee)$-term is also a Hobby-McKenzie term, then by Lemma 9.5 of [10], the variety of meet semilattices does not have such a term. Let $\mathcal{M}$ be a non-trivial variety of modules and, to obtain a contradiction, assume that $t(x_1, \dots, x_n)$ is an $\mathsf{SD}(\vee)$-term of $\mathcal{M}$. We may assume, without loss of generality, that t depends on all of its variables in $\mathcal{M}$.

If we let $\mathbf{R}$ be the ring of $\mathcal{M}$, then there are elements $r_i \in R$ such that

$$\mathcal{M} \models t(x_1, \dots, x_n) = r_1x_1 + r_2x_2 + \dots + r_nx_n.$$

Since t depends on all of its variables in $\mathcal{M}$ then $r_i \neq 0$ for all i. By applying the $\mathsf{SD}(\vee)$-term definition to t and to the set $S = \{x_1, \dots, x_n\}$ it follows that $\mathcal{M}$ satisfies an equation of the form $t(x, x, \dots, x) = t(x, x, \dots, x, y, x, \dots, x)$ where y occurs exactly once on the righthand side, say in the ith position. From the idempotency of t this can be transformed in to the equation

$$x = r_1x + r_2x + \cdots r_{i-1}x + r_iy + r_{i+1}x + \cdots + r_nx$$

and from this we can deduce that the equation $r_iy = 0$ holds in $\mathcal{M}$. This contradicts that $r_i \neq 0$ and so $\mathcal{M}$ cannot have an $\mathsf{SD}(\vee)$-term. •

Corollary 4.4 *Let $\mathcal{V}$ be a variety that has an $\mathsf{SD}(\vee)$-term. Then*

1. *$\mathcal{V}$ is congruence join semidistributive and*
2. *if $\mathcal{V}$ is locally finite then it omits the unary, affine, and semilattice types.*

PROOF: This is immediate from the Proposition and from Theorem 4.1. •

The notion of an $\mathsf{SD}(\vee)$-term was developed while searching for nice conditions that imply omitting the unary, affine, and semilattice types and this in turn was motivated by trying to understand the properties of the class $\mathsf{CSP}(\mathbf{A})$ when $\mathbf{A}$ is a finite algebra that generates a variety that omits the unary, affine, and semilattice types. It is conjectured by Larose and Tesson [14] that for such algebras, $\mathsf{CSP}(\mathbf{A})$ can be solved by an algorithm that runs in non-deterministic logarithmic space.

We do not know if the converse to Corollary 4.4 is true, but at least it is for congruence distributive varieties. We also note that the $\mathcal{V}_n$ terms $t_n(x_0, \ldots, x_{2n})$ from the previous section are $\mathsf{SD}(\vee)$-terms. We make use of the following characterization of congruence distributive varieties due to Jónsson.

Theorem 4.5 (Jónsson) *A variety is congruence distributive if and only if for some $n > 0$ it has terms $p_i(x, y, z)$, $0 \leq i \leq n$ that satisfy the equations*

$$\begin{aligned} p_0(x,y,z) &= x \\ p_n(x,y,z) &= z \\ p_i(x,y,x) &= x \text{ for all } i \\ p_i(x,x,y) &= p_{i+1}(x,x,y) \text{ for all } i \text{ even} \\ p_i(x,y,y) &= p_{i+1}(x,y,y) \text{ for all } i \text{ odd} \end{aligned}$$

Theorem 4.6 *If $\mathcal{V}$ is a congruence distributive variety then it has an $\mathsf{SD}(\vee)$-term.*

PROOF: Suppose that $\mathcal{V}$ has a sequence of ternary terms p_i, for $0 \leq i \leq n$, that satisfy the equations from the previous theorem. Without loss of generality, we may assume that n is even, and so the equation $d_{n-1}(x, y, y) = y$ holds in $\mathcal{V}$. For each i with $1 \leq i < n$ we define inductively a 3^i-ary auxiliary term t_i as follows:

$$\begin{aligned} t_1(x_0, x_1, x_2) &= p_1(x_0, x_1, x_2) \\ t_{i+1}(x_0, \ldots, x_{3^{i+1}-1}) &= p_{i+1}(t_i(x_0, \ldots), t_i(x_{3^i}, \ldots), t_i(x_{2(3^i)}, \ldots)) \end{aligned}$$

Note that the term t_1 is close to being an $\mathsf{SD}(\vee)$-term for $\mathcal{V}$. The only subset of its variables for which the condition from Definition 4.2 fails is $\{x_0\}$. We show by induction on $i < n-1$, that for each term t_i, this is also the case.

Claim: For $1 \leq i < n$, the term $t_i(x_0, \ldots, x_{3^i-1})$ satisfies the condition $(*)_I$ from Definition 4.2 for all subsets I of its variables, except possibly for $I = \{x_0\}$.

As noted, the claim holds for t_1. Assume that it is true for t_i and let I be some subset of the variables of t_{i+1} different from $\{x_0\}$. Denote the first third, the middle third, and the last third of the variables of t_{i+1} by V_0, V_1, and V_2, respectively and let $I_k = I \cap V_k$, for $k = 0$, 1, 2. If $I_1 \neq \emptyset$ then using the identity $p_{i+1}(x, y, x) = x$ it is not hard to see that the identity

$$t_{i+1}(x, x, \ldots, x, y, x, \ldots, x, x) = x$$

holds, where the variable y appears in the jth spot. From this it follows that $(*)_I$ holds for t_{i+1}. Suppose that $I \subseteq V_0 \cup V_2$ and that $I \not\subseteq \{x_0, x_{2(3^i)}\}$. Then either $I_0 \neq \{x_0\}$ or $I_2 \neq \{x_{2(3^i)}\}$, or both. In the former case, we know by induction that $(*)_{I_0}$ holds for t_i. Using the equation that witnesses this, we can extend it to an equation involving t_{i+1} that witnesses $(*)_I$ by assigning the variable x to all x_j for $j \geq 3^i$. Something similar can be done in the case that $I_2 \neq \{x_{2(3^i)}\}$.

To complete the proof of the claim, we need to handle the cases $I = \{x_0, x_{2(3^i)}\}$ and $I = \{x_{2(3^i)}\}$. If i is even, then using the identity $p_i(x,x,y) = p_{i+1}(x,x,y)$, it follows that the equation

$$t_{i+1}(x,\dots,x,y,\dots,y,x,\dots,x,y,\dots,y,x,\dots,x,y,\dots,y) = t_{i+1}(x,\dots,x,y,\dots,y)$$

holds, where on both sides of the equation x occurs $2(3^i)$ times, on the left side in three equal blocks, and y occurs in three equal blocks of size 3^{i-1} on the left side. This equation witnesses $(*)_I$ for both remaining cases. If i is odd, then we can draw a similar conclusion, using the equation $p_i(x,y,y) = p_{i+1}(x,y,y)$.

To conclude the proof of the theorem, we note that by using the equation $p_{n-1}(x,y,y) = y$, the equation

$$t_{n-1}(y,x,x,\dots,x) = t_{n-1}(x,x,\dots,x) = x$$

holds and witnesses $(*)_{\{x_0\}}$. Thus t_{n-1} is an $\mathsf{SD}(\vee)$-term for $\mathcal{V}$. •

We conclude with an even stronger term condition that may also be useful for characterizing varieties that omit the unary, affine, and semilattice types and for analyzing the complexity of $\mathsf{CSP}(\mathbf{A})$ for certain algebras $\mathbf{A}$. We observe that any term $t(x_1,\dots,x_n)$ that satisfies the following is an $\mathsf{SD}(\vee)$-term:

> there is a partition $(I_i : 1 \leq i \leq m)$ of $\{1,2,\dots,n\}$ such that for any $i \leq m$ and $j \in I_i$ the equation $t(a_1,\dots,a_n) = t(b_1,\dots,b_n)$ holds, where
>
> - $a_k = b_k = x$ for all $k \in I_r$ if $r < i$,
> - $a_k = b_k = y$ for all $k \in I_r$ if $r > i$,
> - $a_k = x$ for all $k \in I_i$,
> - $b_k = x$ for all $k \in I_i \setminus \{j\}$, and
> - $b_j = y$.

Problem: Determine if every (locally finite) congruence join semidistributive variety has a term that satisfies this condition.

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