ROBUSTNESS OF CONGRUENCE SEMIDISTRIBUTIVITY

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In this note we prove the following theorem, showing the property of congruence semidistributivity is robust in the sense of McKenzie.

Theorem 1. *If $\mathcal{V}$ and $\mathcal{W}$ are idempotent, congruence join semidistributive varieties of the same type, then $\boldsymbol{V}(\mathcal{V} \circ \mathcal{W})$ is also congruence join semidistributive. ($\mathcal{V} \circ \mathcal{W}$ denotes the Maltsev product.)*

We use the characterization of join semidistributivity from [1].

Theorem 2. *A variety $\mathcal{V}$ is congruence semidistributive if and only if it has an idempotent term $t(x_1, \dots, x_n)$ such that for each k there are sequences $\mathbf{w}$ and $\mathbf{w}'$ of x's and y's, both of length $n - k$, such that $\mathcal{V}$ satisfies*

$$t(x, \dots, x, x, \mathbf{w}) \approx t(x, \dots, x, y, \mathbf{w}'). \tag{1}$$

In matrix form:

$$t \begin{bmatrix} x & & & & \\ x & x & & & \\ x & x & x & & \\ & & & \ddots & \\ x & x & x & \cdots & x \end{bmatrix} \approx t \begin{bmatrix} y & & & & \\ x & y & & & \\ x & x & y & & \\ & & & \ddots & \\ x & x & x & \cdots & y \end{bmatrix}$$

Proof of Theorem 1. Let $s(x_1, \dots, x_m)$ be a term as in Theorem 2 that works for $\mathcal{W}$ and let $r(x_1, \dots, x_n)$ be a term that works for $\mathcal{V}$. Define an mn-ary term t by

$$\begin{aligned} &t(x_1, \dots, x_{mn}) \\ &\qquad = r(s(x_1, x_{n+1}, \dots), s(x_2, x_{n+2}, \dots), \dots, s(x_n, x_{2n}, \dots, x_{mn})) \end{aligned} \tag{2}$$

Note the variables are distributed in the right side using the transpose of the usual way.

Let θ be a congruence on an idempostent algebra $\mathbf{A}$ and suppose $\mathbf{A}/\theta \in \mathcal{W}$ and that each block of θ lies in $\mathcal{V}$. We wish to show that, for $k = 1, \dots, mn$, there are $\mathbf{w}$ and $\mathbf{w}'$ so that (1) holds for the t of (2).

Let i and j be such that the variable x_k in (2) occurs in the i^{th} place of the $s(\dots)$ which is in the j^{th} place of r. There are vectors $\mathbf{u}$ and $\mathbf{u}'$ of length $m - i$ such that $\mathbf{A}/\theta$ satisfies

$$s(x, \dots, x, x, \mathbf{u}) \approx s(x, \dots, x, y, \mathbf{u}') \tag{3}$$

Let $\bar{x} = s(x, \dots, x, x, \mathbf{u})$ and $\bar{y} = s(x, \dots, x, y, \mathbf{u}')$. Our goal is to satisfy (1) for k. That equation specifies the substitution $x_\ell \mapsto x$, $\ell = 1, \dots, k-1$ on both sides and $x_k \mapsto x$ on the left and $x_k \mapsto y$ on the right. So we need to specify the other variables (as either x or y) on both sides so that $\mathbf{A}$ satisfies (1). At the s which lies in the j^{th} position of r we use the substitution as in (3) so that the left side has $\bar{x}$ and the right side has $\bar{y}$ at the j^{th} position of r. For the positions to the left of this, we choose the unassigned variable so that all of them (on both sides of the equation) are $\bar{x}$. For those s's to the right of position j of r, only the first $i-1$ variables have been specified (all with value x). This means that it is possible to choose the unassigned variables to give either $\bar{x}$ or $\bar{y}$, independently.

Now the i^{th} equation for r has the form

$$r(x, \dots, x, x, \mathbf{v}) \approx r(x, \dots, x, y, \mathbf{v}')$$

By the argument above, we can assign the variables of t so that the k^{th} equation has the form

$$r(\bar{x}, \dots, \bar{x}, \bar{x}, \bar{\mathbf{v}}) \approx r(\bar{x}, \dots, \bar{x}, \bar{y}, \bar{\mathbf{v}}') \tag{4}$$

where $\bar{\mathbf{v}}$ is $\mathbf{v}$ with the substitution $x \mapsto \bar{x}$ and $y \mapsto \bar{y}$, and similarly for $\mathbf{v}'$.

Since $\bar{x}\ \theta\ \bar{y}$ under any substition into $\mathbf{A}$, $\bar{x}$ and $\bar{y}$ lie in the same block. Thus (4) holds in $\mathbf{A}$ for any substitution. This shows that (1) hold in $\mathbf{A}$, proving the theorem. □

References

[1] R. Freese, M. Kozik, A. Krokhin, B. Larose, M. Valeriote, and R. Willard, *Some new characterizations of some old Maltsev conditions.*