A NOTE ON CONGRUENCE JOIN SEMIDISTRIBUTIVITY

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In this note we give a Maltsev condition for congruence join semidistributivity, $(\mathrm{SD}_\vee)$, which is a slight variant of the usual one given in [2] and [4] and use this to answer a question in [5].

Lemma 1. *If $d = a \vee b = a \vee c = b \vee c$ in a lattice satisfying $(\mathrm{SD}_\vee)$, then*

$$d = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c).$$

Proof. $(\mathrm{SD}_\vee)$ implies $d = a \vee (b \wedge c)$ and $d = b \vee (a \wedge c)$. So

$$d = [(b \wedge c) \vee (a \wedge c)] \vee a = [(b \wedge c) \vee (a \wedge c)] \vee b$$

Applying $(\mathrm{SD}_\vee)$ gives the lemma. □

This lemma is related to a result of Jónsson and Kiefer [3]; see also Theorem 1.21 of [1].

Theorem 2. *The following are equivalent for a variety $\mathcal{V}$:*

(1) *$\mathcal{V}$ is congruence join semidistributive.*

(2) *If $\mathbf{A} \in \mathcal{V}$ and α, β and $\gamma \in \mathrm{Con}(\mathbf{A})$, then*

$$\alpha \cap (\beta \circ \gamma) \subseteq (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \vee (\beta \wedge \gamma).$$

(3) *There is a positive integer k and ternary terms $d_0, \dots, d_k$ such that $\mathcal{V}$ satisfies the following equations:*
   - (a) $d_0(x, y, z) \approx x$;
   - (b) $d_i(x, y, y) \approx d_{i+1}(x, y, y)$ *if* $i \equiv 0$ *or* $1 \mod 3$;
   - (c) $d_i(x, y, x) \approx d_{i+1}(x, y, x)$ *if* $i \equiv 0$ *or* $2 \mod 3$;
   - (d) $d_i(x, x, y) \approx d_{i+1}(x, x, y)$ *if* $i \equiv 1$ *or* $2 \mod 3$;
   - (e) $d_k(x, y, z) \approx z$;

Proof. Assume (1) and let $(a, c) \in \alpha \cap (\beta \circ \gamma)$. Then there is an element b such that $a \mathrel{\beta} b$ and $b \mathrel{\gamma} c$. Let $\alpha' = \mathrm{Cg}^{\mathbf{A}}(a, c)$, $\beta' = \mathrm{Cg}^{\mathbf{A}}(a, b)$ and $\gamma' = \mathrm{Cg}^{\mathbf{A}}(b, c)$. Clearly every pair from $\{\alpha', \beta', \gamma'\}$ joins to $\mathrm{Cg}^{\mathbf{A}}(a, b, c)$. By the lemma this implies

$$(\alpha' \wedge \beta') \vee (\alpha' \wedge \gamma') \vee (\beta' \wedge \gamma') = \mathrm{Cg}^{\mathbf{A}}(a, b, c)$$

Since $\alpha' \leq \alpha$, etc., this easily implies that (2) holds.

(3) follows from (2) by the standard Maltsev-Pixley-Wille argument and, since the left side, $\alpha \cap (\beta \circ \gamma)$, does not have any joins, (3) implies (2) by a standard argument. So we have (1) implies (2) and (2) and (3) are equivalent.

Let (2′) be like (2) but with the inclusion changed to

$$\alpha \cap (\beta \circ \gamma) \subseteq \beta \vee (\alpha \wedge \gamma).$$

Now (2) implies (2′) because the right side of the inclusion in (2′) contains the right side of the inclusion of (2). But (2′) is equivalent to (1) by Theorem 8.14 of [4], completing the proof.

Alternately, if (3′) is the Maltsev condition for ($\mathrm{SD}_\vee$) given in [2] and [4], then it is easy to see that (3) implies (3′) just be dropping some of the d_i's. So (1) $\implies$ (2) $\implies$ (3) $\implies$ (3′) $\implies$ (1). □

An SD($\vee$)-term is defined in [5] to be an idempotent term $t(x_0, \ldots, x_{n-1})$ of a variety $\mathcal{V}$ such that for all nonempty subsets S of $\{x_0, \ldots, x_{n-1}\}$

> $\mathcal{V}$ satisfies an equation in the variables $\{x, y\}$ of the form $t(a_0, \ldots, a_{n-1}) \approx t(b_0, \ldots, b_{n-1})$, where $a_i = x$ for all i such that $x_i \in S$, and $b_i = x$ for all i such that $x_i \in S$ except for exactly one such i (for which $b_i = y$, of course).

Let $(*)_S$ denote the displayed condition.

Theorem 3. *A variety $\mathcal{V}$ is congruence join semidistributive if and only if it has an* SD($\vee$)*-term.*

Proof. Suppose $\mathcal{V}$ is congruence join semidistributive and so has terms as in condition (3) of Theorem 2. Without loss of generality we may assume k is 1 or 2 mod 3 so that $d_{k-1}(x, y, y) \approx y$ holds in $\mathcal{V}$. Define terms $t_1, \ldots, t_{k-1}$ inductively by:

$$t_1(x_0, x_1, x_2) = d_1(x_0, x_1, x_2)$$
$$t_{i+1}(x_0, \ldots, x_{3^{i+1}-1}) = d_{i+1}(t_i(x_0, \ldots), t_i(x_{3^i}, \ldots), t_i(x_{2\cdot 3^i}, \ldots))$$

Claim 4. *For $1 \le i < k$ the term $t_i(x_0, \ldots, x_{3^i-1})$ satisfies $(*)_I$ for all nonempty subsets I of its variables except possibly $I = \{x_0\}$.*

Since $t_1(x, x, x) \approx t_1(x, y, y)$ and $t_1(x, x, x) \approx t_1(x, y, x)$, the claim holds for $i = 1$. Suppose $I \nsubseteq \{x_0, x_{3^i}, x_{2\cdot 3^i}\}$. Then there is a j such that $x_j \in I$ but not in $\{x_0, x_{3^i}, x_{2\cdot 3^i}\}$. So either $0 < j < 3^i$ or $3^i < j < 2 \cdot 3^i$ or $2 \cdot 3^i < j < 3^{i+1}$. The arguments for the three cases are similar, so we shall assume $3^i < j < 2 \cdot 3^i$ and let $I_1 = I \cap \{x_{3^i}, \ldots, x_{2\cdot 3^i - 1}\}$. By induction $(*)_{I_1}$ holds for $t_i(x_{3^i}, \ldots, x_{2\cdot 3^i-1})$ so there is an equation witnessing this. We can extend this to a witness that t_{i+1} satisfies $(*)_I$ by setting the first third and last third of the variables to x.

So we may assume $I \subseteq \{x_0, x_{3^i}, x_{2\cdot 3^i}\}$. Let $\bar{x}$ be a string of 3^{i-1} x's and $\bar{\bar{x}}$ be a string of 3^i x's and similarly for y. Then

$$\begin{aligned}
t_{i+1}(\bar{x},\bar{y},\bar{y},\bar{x},\bar{y},\bar{y},\bar{x},\bar{y},\bar{y}) &\approx t_{i+1}(\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{y}}) && i = 0 \text{ or } 1 \pmod 3\\
t_{i+1}(\bar{x},\bar{y},\bar{x},\bar{x},\bar{y},\bar{x},\bar{x},\bar{y},\bar{x}) &\approx t_{i+1}(\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{x}}) && i = 0 \text{ or } 2 \pmod 3\\
t_{i+1}(\bar{x},\bar{x},\bar{y},\bar{x},\bar{x},\bar{y},\bar{x},\bar{x},\bar{y}) &\approx t_{i+1}(\bar{\bar{x}},\bar{\bar{x}},\bar{\bar{y}}) && i = 1 \text{ or } 2 \pmod 3
\end{aligned}$$

For example both sides of the first equation can be simplified to $d_i(x, y, y)$.

To complete the proof of the claim we need to verify $(*)_I$ for all nonempty subsets I of $\{x_0, x_{3^i}, x_{2\cdot 3^i}\}$ except $\{x_0\}$. Using the above equations, this is straightforward.

Using $d_{k-1}(x, y, y) \approx y$ one can show

$$t_{k-1}(y, x, \ldots, x) \approx t_{k-1}(x, x, \ldots, x) \approx x$$

which witnesses $(*)_{\{x_0\}}$. Thus t_{k-1} is an SD($\vee$)-term.

The other direction is proved in [5]. □

References

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