

### Jónsson's Lemma

#### 1. A finite version

*Theorem.* (Foster) Let  $A$  be a finite algebra such that  $\text{Var}(A)$  is congruence-distributive. Let  $B \in \text{Var}(A)$  be finite and subdirectly irreducible. Then  $B \in \mathbf{HS}(A)$ .

*Corollary.* Under the same hypotheses,  $|B| \leq |A|$ , and if  $|B| = |A|$  then  $B \cong A$ .

*Example.* Each of the lattices  $M_3, N_5$  satisfies a law that fails in the other.

*Proof of the theorem:*  $\text{Var}(A) = \mathbf{HSP}(A)$ , so represent  $B$  as a homomorphic image of a subalgebra  $C$  of  $A \times \cdots \times A$ :  $C \subseteq A \times \cdots \times A$  and  $\phi : C \rightarrow B$  (a surjection). Here we know a finite product will do since  $B$  is the image of a free algebra  $\text{Var}_A(n)$ , where  $n = |B|$ , and such a free algebra can be constructed by the table method. See the left-hand side of Figure 1.

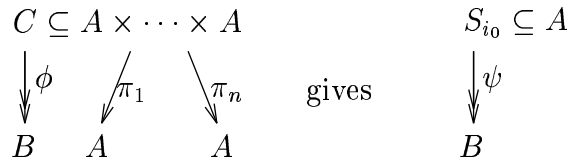


Figure 1: Mappings for the Theorem of §1.

Focus on  $\text{Con}(C)$ . One of its elements is  $\ker \phi$ , which by the Correspondence Theorem is meet-irreducible. Some other elements are the kernels of the coordinate projections restricted to  $C$ :  $\ker(\pi_i|_C)$ . Of course  $\pi_i|_C$  may not map  $C$  onto  $A$ ; its image is some subalgebra  $S_i$  of  $A$ .

Observe that

$$\bigcap_i \ker(\pi_i|_C) = 0 \leq \ker \phi.$$

Recall that in a distributive lattice, a meet-irreducible element is meet-prime. Therefore  $\ker(\pi_{i_0}|_C) \leq \ker \phi$  for some  $i_0$ . This says that  $\pi_{i_0}(a) = \pi_{i_0}(a') \Rightarrow \phi(a) = \phi(a')$ . Therefore a well defined map  $\psi$  of the image of  $S_{i_0}$  onto  $B$  is obtained by setting  $\psi(\pi_{i_0}(a)) = \phi(a)$ . This map is the desired homomorphism showing that  $B \in \mathbf{HS}(A)$ . See the right-hand side of Figure 1.  $\square$

## 2. Ultrafilters

Consider the set  $I = \omega = \{0, 1, 2, \dots\}$ , the lattice  $\text{Pow}(I)$ , and its ideals and dual ideals (filters). A principal ideal consists of the family of all subsets of some given set. Some examples to think about:

- The principal ideal generated by  $I \setminus \{k\}$  is maximal, for each  $k$ .
- Its complement is the principal dual ideal (“principal ultrafilter”) consisting of all subsets containing  $\{k\}$ .
- The ideal  $I_0$  of all finite subsets is not principal.
- However, the ideal  $I_0$  of all finite subsets is the intersection of maximal ideals (as is any ideal). These are the *nonprincipal* maximal ideals. There are  $2^{2^{\aleph_0}}$  of them, but it is impossible to give even one explicitly!

Given a maximal ideal, we think of its members as “small” subsets of  $I$ . What is a “large” subset? There are two possible definitions:

- (1) A large subset is a subset that is not small;
- (2) a large subset is the complement of a small subset.

But these two definitions are equivalent! Recall that for a maximal ideal of a Boolean lattice, for each  $x$  exactly one of  $x$  or  $x'$  is in the ideal.

*Question.* For the principal maximal ideal generated by  $I \setminus \{k\}$ , which subsets of  $I$  are small and which large? (It is as if only  $k$  counts for largeness.)

To summarize,

1. Every subset of  $I$  is either large or small (not both).
2. The empty set is small. In fact, if the maximal ideal is nonprincipal, then any finite subset is small.
3.  $I$  itself is large.
4. The union of two small subsets is small.
5. The intersection of two large subsets is large.
6. A subset of a small subset is small.
7. A superset of a large subset is large.
8. The small sets form a maximal ideal.
9. The large sets form an ultrafilter.

### 3. Ultraproducts

An “ultraproduct” of algebras is their direct product modulo an congruence relation constructed from a nonprincipal ultrafilter. The congruence relation tends to collapse the product down to something that looks like a “generic” copy of the individual algebras, reflecting whatever features they have in common.

The construction is set-theoretic and actually works for sets with relations as well as for algebras. In detail:

*Definition.* Let  $I$  be an infinite index set. Let algebras  $A_i, i \in I$  be given. Choose a nonprincipal ultrafilter  $\mathcal{U}$  on  $I$ . On the direct product  $\prod_{i \in I} A_i$ , define an relation  $\equiv$  by saying  $\mathbf{a} \equiv \mathbf{b}$  when  $\mathbf{a}$  and  $\mathbf{b}$  agree on a large set of indices. The *ultraproduct* of the  $A_i$  is the direct product modulo  $\equiv$ :

$$A^* = (\prod_{i \in I} A_i) / \equiv, \text{ or more simply } A^* = \prod_{i \in I} A_i / \mathcal{U}.$$

There are several things to consider here:

- Does the phrase “agree on a large set of indices” mean that there is *some* large set  $J \subseteq I$  of indices such that  $a_j = b_j$  for all  $j \in J$ , or that the set of *all*  $i \in I$  with  $a_i = b_i$  is large? By the properties of large sets, it doesn’t matter; the meanings are the same.
- It must be checked that  $\equiv$  is an equivalence relation. This follows from the properties of large sets.
- We say “the” ultraproduct even though the result does depend on the choice of  $\mathcal{U}$ .

Ultraproducts have some startling properties:

1. Any  $n$ -ary relation common to the  $A_i$  has a reasonable definition on their ultraproduct.
2. Any first-order sentence true in the  $A_i$  is true in their ultraproduct. (This extends to first-order formulas.)
3. An ultraproduct of fields is a field. (Why?)
4. The ultraproduct is unchanged if finitely many factors are omitted. (Why?)
5. If all the  $A_i$  are finite and isomorphic, then  $A^*$  is a copy of the same algebra. (Why?)

*Examples.*

- (a) The ultraproduct of countably many copies of the field  $\mathbf{R}$  of reals is the field  $\mathbf{R}^*$  of “nonstandard reals”. It is possible to do calculus using “infinitesimals” in  $\mathbf{R}^*$ .
- (b) The ultraproduct of countably many copies of the ring  $\mathbf{Z}$  of integers is the ring  $\mathbf{Z}^*$  of “nonstandard integers”. Some of them are “infinite”.
- (c) The ultraproduct  $\mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{Z}_5 \times \cdots / \mathcal{U}$  is a field of characteristic 0.
- (d) The ultraproduct of chains  $\mathbf{1} \times \mathbf{2} \times \mathbf{3} \times \cdots / \mathcal{U}$  is an infinite chain. (What does it look like?)

## 4. Jónsson’s Lemma

“Jónsson’s Lemma” would be called a theorem by most people, but it was called a lemma in the original paper and the name has stuck.

For a class  $\mathcal{K}$  of similar algebras, let  $\mathbf{U}(\mathcal{K})$  denote the class of algebras isomorphic to ultraproducts of algebras in  $\mathcal{K}$ <sup>1</sup>.

*Theorem.* (Jónsson’s Lemma) Let  $\mathcal{K}$  be a class of similar algebras such that  $\text{Var}(\mathcal{K})$  is congruence-distributive. If  $B \in \text{Var}(\mathcal{K})$  is subdirectly irreducible, then  $B \in \mathbf{HSU}(\mathcal{K})$ .

*Corollary.* For a finite algebra  $A$ , if  $\text{Var}(A)$  is congruence-distributive, then for each subdirectly irreducible algebra  $B \in \text{Var}(A)$  we have  $B \in \mathbf{HS}(A)$ .

Notice that this Corollary is a little stronger than the Theorem of §1, since it is not assumed to start with that  $B$  is finite. The conclusion is the same.

## 5. Problems

**Problem G-1.** How can we be sure that an ultraproduct of chains is a chain?

**Problem G-2.** Prove the Corollary of §4 from Jónsson’s Lemma.

**Problem G-3.** Let  $\mathbf{F}_4$  be the Galois field of 4 elements. Find all the SI members of  $\text{Var}(\mathbf{F}_4)$ , up to isomorphism.

---

<sup>1</sup>Most authors write  $\mathbf{P}_U$ , following Jónsson, and some omit the use of isomorphic copies.