Jónsson's Lemma

1. A finite version

Theorem. (Foster) Let A be a finite algebra such that $\mathrm{Var}(A)$ is congruence-distributive. Let $B \in \mathrm{Var}(A)$ be finite and subdirectly irreducible. Then $B \in \mathbf{HS}(A)$.

Corollary. Under the same hypotheses, $|B| \leq |A|$, and if $|B| = |A|$ then $B \cong A$.

Example. Each of the lattices M_3, N_5 satisfies a law that fails in the other.

Proof of the theorem: $\mathrm{Var}(A) = \mathbf{HSP}(A)$, so represent B as a homomorphic image of a subalgebra C of $A \times \cdots \times A$: $C \subseteq A \times \cdots \times A$ and $\phi : C \to B$ (a surjection). Here we know a finite product will do since B is the image of a free algebra $\mathrm{Var}_A(n)$, where $n = |B|$, and such a free algebra can be constructed by the table method. See the left-hand side of Figure **??**.

$C \subseteq A \times \cdots \times A$				$S_{i_0} \subseteq A$
ϕ	π_1	π_n	gives	ψ
B	A	A		B

Figure 1: Mappings for the Theorem of §1.

Focus on $\mathrm{Con}(C)$. One of its elements is $\ker\phi$, which by the Correspondence Theorem is meet-irreducible. Some other elements are the kernels of the coordinate projections restricted to C: $\ker(\pi_i|_C)$. Of course $\pi_i|_C$ may not map C onto A; its image is some subalgebra S_i of A.

Observe that

$$\cap_i \ker(\pi_i|_C) = 0 \leq \ker\phi.$$

Recall that in a distributive lattice, a meet-irreducible element is meet-prime. Therefore $\ker(\pi_{i_0}|_C) \leq \ker\phi$ for some i_0. This says that $\pi_{i_0}(a) = \pi_{i_0}(a') \Rightarrow \phi(a) = \phi(a')$. Therefore a well defined map ψ of the image of S_{i_0} onto B is obtained by setting $\psi(\pi_{i_0}(a)) = \phi(a)$. This map is the desired homomorphism showing that $B \in \mathbf{HS}(A)$. See the right-hand side of Figure **??**. $\square$

2. Ultrafilters

Consider the set $I = \omega = \{0, 1, 2, \dots\}$, the lattice $\text{Pow}(I)$, and its ideals and dual ideals (filters). A principal ideal consists of the family of all subsets of some given set. Some examples to think about:

- The principal ideal generated by $I \setminus \{k\}$ is maximal, for each k.
- Its complement is the principal dual ideal ("principal ultrafilter") consisting of all subsets containing $\{k\}$.
- The ideal I_0 of all finite subsets is not principal.
- However, the ideal I_0 of all finite subsets is the intersection of maximal ideals (as is any ideal). These are the *nonprincipal* maximal ideals. There are $2^{2^{\aleph_0}}$ of them, but it is impossible to give even one explicitly!

Given a maximal ideal, we think of its members as "small" subsets of I. What is a "large" subset? There are two possible definitions:

(1) A large subset is a subset that is not small;

(2) a large subset is the complement of a small subset.

But these two definitions are equivalent! Recall that for a maximal ideal of a Boolean lattice, for each x exactly one of x or x' is in the ideal.

Question. For the principal maximal ideal generated by $I \setminus \{k\}$, which subsets of I are small and which large? (It is as if only k counts for largeness.)

To summarize,

1. Every subset of I is either large or small (not both).
2. The empty set is small. In fact, if the maximal ideal is nonprincipal, then any finite subset is small.
3. I itself is large.
4. The union of two small subsets is small.
5. The intersection of two large subsets is large.
6. A subset of a small subset is small.
7. A superset of a large subset is large.
8. The small sets form a maximal ideal.
9. The large sets form an ultrafilter.

3. Ultraproducts

An "ultraproduct" of algebras is their direct product modulo an congruence relation constructed from a nonprincipal ultrafilter. The congruence relation tends to collapse the product down to something that looks like a "generic" copy of the individual algebras, reflecting whatever features they have in common.

The construction is set-theoretic and actually works for sets with relations as well as for algebras. In detail:

Definition. Let I be an infinite index set. Let algebras $A_i, i \in I$ be given. Choose a nonprincipal ultrafilter $\mathcal{U}$ on I. On the direct product $\prod_{i \in I} A_i$, define an relation $\equiv$ by saying $\mathbf{a} \equiv \mathbf{b}$ when $\mathbf{a}$ and $\mathbf{b}$ agree on a large set of indices. The *ultraproduct* of the A_i is the direct product modulo $\equiv$:

$$A^* = (\textstyle\prod_{i \in I} A_i)/\equiv, \text{ or more simply } A^* = \prod_{i \in I} A_i/\mathcal{U}.$$

There are several things to consider here:

- Does the phrase "agree on a large set of indices" mean that there is *some* large set $J \subseteq I$ of indices such that $a_j = b_j$ for all $j \in J$, or that the set of *all* $i \in I$ with $a_i = b_i$ is large? By the properties of large sets, it doesn't matter; the meanings are the same.
- It must be checked that $\equiv$ is an equivalence relation. This follows from the properties of large sets.
- We say "the" ultraproduct even though the result does depend on the choice of $\mathcal{U}$.

Ultraproducts have some startling properties:

1. Any n-ary relation common to the A_i has a reasonable definition on their ultraproduct.
2. Any first-order sentence true in the A_i is true in their ultraproduct. (This extends to first-order formulas.)
3. An ultraproduct of fields is a field. (Why?)
4. The ultraproduct is unchanged if finitely many factors are omitted. (Why?)
5. If all the A_i are finite and isomorphic, then A^* is a copy of the same algebra. (Why?)

Examples.

(a) The ultraproduct of countably many copies of the field $\mathbf{R}$ of reals is the field $\mathbf{R}^*$ of "nonstandard reals". It is possible to do calculus using "infinitesimals" in $\mathbf{R}^*$.

(b) The ultraproduct of countably many copies of the ring $\mathbf{Z}$ of integers is the ring $\mathbf{Z}^*$ of "nonstandard integers". Some of them are "infinite".

(c) The ultraproduct $\mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{Z}_5 \times \cdots / \mathcal{U}$ is a field of characteristic 0.

(d) The ultraproduct of chains $\mathbf{1} \times \mathbf{2} \times \mathbf{3} \times \cdots / \mathcal{U}$ is an infinite chain. (What does it look like?)

4. Jónsson's Lemma

"Jónsson's Lemma" would be called a theorem by most people, but it was called a lemma in the original paper and the name has stuck.

For a class $\mathcal{K}$ of similar algebras, let $\mathbf{U}(\mathcal{K})$ denote the class of algebras isomorphic to ultraproducts of algebras in $\mathcal{K}$ [1].

Theorem. (Jónsson's Lemma) Let $\mathcal{K}$ be a class of similar algebras such that $\mathrm{Var}(\mathcal{K})$ is congruence-distributive. If $B \in \mathrm{Var}(\mathcal{K})$ is subdirectly irreducible, then $B \in \mathbf{HSU}(\mathcal{K})$.

Corollary. For a finite algebra A, if $\mathrm{Var}(A)$ is congruence-distributive, then for each subdirectly irreducible algebra $B \in \mathrm{Var}(A)$ we have $B \in \mathbf{HS}(A)$.

Notice that this Corollary is a little stronger than the Theorem of §1, since it is not assumed to start with that B is finite. The conclusion is the same.

5. Problems

Problem G-1. How can we be sure that an ultraproduct of chains is a chain?

Problem G-2. Prove the Corollary of §**??** from Jónsson's Lemma.

Problem G-3. Let $\mathbf{F}_4$ be the Galois field of 4 elements. Find all the SI members of $\mathrm{Var}(\mathbf{F}_4)$, up to isomorphism.

[1] Most authors write $\mathbf{P}_U$, following Jónsson, and some omit the use of isomorphic copies.