

**NOTES ON CONGRUENCE
 n -PERMUTABILITY AND SEMIDISTRIBUTIVITY**

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ABSTRACT. In [1] T. Dent, K. Kearnes and Á. Szendrei define the *derivative*, Σ' , of a set of equations Σ and show, for idempotent Σ , that Σ implies congruence modularity if Σ' is inconsistent ($\Sigma' \models x \approx y$). In this paper we investigate other types of derivatives that give similar results for congruence n -permutable for some n , and for congruence semidistributivity.

In a recent paper [1] T. Dent, K. Kearnes and Á. Szendrei study Maltsev conditions which imply congruence modularity from the point of view of the equations. Given a set of equations Σ they define the *derivative* Σ' of Σ . (The definition is given below.) $\Sigma' \supseteq \Sigma$ and if Σ is idempotent then, if Σ' is inconsistent (that is, $\Sigma' \models x \approx y$), then any variety \mathcal{V} that *realizes* Σ (each function symbol in Σ can be interpreted as a term of \mathcal{V} such that the equations of Σ are satisfied) is congruence modular. While the converse is not true in general, they show that it is true if Σ is a set of *linear*, idempotent equations. So in particular if Σ consists of Day's equations (for a fixed n) or Gumm's equations, then Σ' is inconsistent.

They use these results to prove interesting new results and give easy proofs of several existing theorems. One nice example: one of the equations in Day's characterization of congruence modularity involves three variables:

$$m_i(x, u, u, y) \approx m_{i+1}(x, u, u, y) \quad \text{for } i \text{ odd}$$

J. B. Nation wondered if there is a two-variable condition implying congruence modularity. In [9] he showed that this is the case.

Using their results, the authors of [1] show that the above equation can be replaced by either

$$m_i(x, x, x, y) \approx m_{i+1}(x, x, x, y) \quad \text{for } i \text{ odd}$$

or

$$m_i(x, y, y, y) \approx m_{i+1}(x, y, y, y) \quad \text{for } i \text{ odd}$$

and the resulting set of identities still implies congruence modularity.

The principal results of [1] for congruence modularity (CM) are summarized in the next theorem. Σ *idempotent* means $\Sigma \models f(x, \dots, x) \approx x$ for all operation symbols f occurring in Σ ; it is *linear* if there is no nested composition in the terms occurring in Σ . If $\Sigma \models x \approx f(\mathbf{w})$, where \mathbf{w} is a sequence of not necessarily distinct variables, then f is *weakly independent* of its i^{th} place for each i with $w_i \neq x$. (So a Maltsev term is weakly independent of

all of its places.) The *derivative*, Σ' , is the augmentation of Σ by equations that say that f is independent of its i^{th} place whenever Σ implies f is weakly independent of its i^{th} place.

Theorem 1. *Let Σ be an idempotent set of equations. Then*

- (1) *If Σ' is inconsistent then any variety that realizes Σ is congruence modular (CM).*
- (2) *If \mathcal{V} is a CM variety, then \mathcal{V} realizes some Σ such that Σ' is inconsistent. (The Day terms work.)*
- (3) *The converse of the first statement is not true in general but it is true if Σ is linear.*
- (4) *For a finite linear, idempotent Σ one can effectively decide if Σ implies CM.*

The authors also show this theorem remains true if “congruence modularity” is replaced by “satisfies some nontrivial congruence identity” and “ Σ' inconsistent” is replaced by “some iterated derivative of Σ is inconsistent.”

For n -permutability we introduce another derivative, the order derivative, denoted Σ^+ . We show the theorem above remains true if “congruence modularity” is replaced by “congruence n -permutable for some n ” and “ Σ' inconsistent” is replaced by “some iterated order derivative of Σ is inconsistent.”

For semidistributivity we define the weak derivative, denoted Σ^* . In this case the analogs of conditions (1) and (2) hold, but (the second part of) (3) fails. Nevertheless we are able to establish (4), both for semidistributivity and meet semidistributivity.

Part (4) of the above theorem and its variants show that for several properties one can effectively decide if a linear, idempotent Σ implies the property; that is every variety that realizes Σ has the property. A list of such properties is given at the end of the paper. This decidability contrasts the general result of G. McNulty [8] that without linearity all of these problems are undecidable.

1. n -PERMUTABILITY

A variety \mathcal{V} is congruence n -permutable if for every $\mathbf{A} \in \mathcal{V}$ and every pair of congruences α and β , $\alpha \circ_n \beta = \beta \circ_n \alpha$, where $\alpha \circ_n \beta = \alpha \circ \beta \circ \alpha \circ \beta \circ \dots$ is the n -fold relational product (there are $n - 1$ occurrences of \circ).

Theorem 2. *The following are equivalent for a variety \mathcal{V} and an integer n at least 2.*

- (1) *\mathcal{V} is congruence n -permutable.*
- (2) *There are 3-place terms p_0, \dots, p_n such that the following hold in \mathcal{V}*

$$p_0(x, y, z) \approx x, \text{ and } p_n(x, y, z) \approx z$$

$$p_i(x, x, y) \approx p_{i+1}(x, y, y) \text{ for all } i.$$

- (3) The subalgebra of $\mathbf{F}_{\mathcal{V}}^2(x, y)$ generated by (x, x) , (x, y) and (y, y) contains elements (a_i, b_i) , $i = 0, \dots, n$, with $(y, y) = (a_0, b_0)$, $(x, x) = (a_n, b_n)$, and $b_i = a_{i+1}$.

Proof. The first two are equivalent by the result of Hagemann and Mitschke [3]. They are equivalent to the third by a standard argument: if (2) holds, let $a_i = p_{n-i}(x, x, y)$ and $b_i = p_{n-i}(x, y, y)$. Then $(a_i, b_i) = p_{n-i}((x, x), (x, y), (y, y))$ and the properties of (3) hold. Conversely if (3) holds the terms giving the (a_i, b_i) 's, indexed backwards, satisfy (2). \square

Theorem 3. *The following are equivalent for an idempotent variety \mathcal{V} .*

- (1) \mathcal{V} is not congruence n -permutable for any n .
- (2) \mathcal{V} contains a nontrivial member \mathbf{A} which has a bounded compatible order (one having a least and greatest element).

Proof. Assume \mathcal{V} is not n -permutable for any n . Let \mathbf{F} be the free \mathcal{V} algebra freely generated by x and y . Let \mathbf{R} be the subalgebra of \mathbf{F}^2 generated by $\{(x, x), (x, y), (y, y)\}$ and let \mathbf{T} be the transitive closure of \mathbf{R} . \mathbf{T} is also a subalgebra of \mathbf{F}^2 and is reflexive and transitive so is a compatible quasiorder on \mathbf{F} . Clearly $(x, y) \in T$.

Suppose $(y, x) \in T$. Then there are elements $w_j \in F$, $j = 0, \dots, n$, such that $w_0 = y$, $w_n = x$ and $(w_j, w_{j+1}) \in R$, $j < n$. Thus

$$(y, y), (y, w_1), (w_1, w_2), \dots, (w_{n-1}, x), (x, x)$$

are all in $\text{Sg}_{\mathbf{F}^2}((x, x), (x, y), (y, y))$. By (3) of Theorem 2, \mathcal{V} is n -permutable, a contradiction. Thus there is a compatible quasiordering \leq on \mathbf{F} with $x \leq y$ and $y \not\leq x$.

Let

$$\theta = \{(a, b) \in \mathbf{F}^2 : a \leq b \text{ and } b \leq a\}$$

be the equivalence relation associated with \leq . Since \leq is compatible, θ is a congruence. Let $\mathbf{A} = \mathbf{F}/\theta$. \mathbf{A} is nontrivial since $(x, y) \notin \theta$. And \mathbf{A} has a compatible ordering which we also denote \leq . Using that \mathcal{V} is idempotent, it is easy to see that $0 = x/\theta$ is the least element and $1 = y/\theta$ is the greatest element in this ordering. This proves (1) \implies (2).

For the other implication suppose $\mathbf{A} \in \mathcal{V}$ is nontrivial and has a compatible order with 0 and 1. If \mathcal{V} has Hagemann-Mitschke terms then

$$1 = p_0^{\mathbf{A}}(1, 1, 0) = p_1^{\mathbf{A}}(1, 0, 0) \leq p_1^{\mathbf{A}}(1, 1, 0) = \dots = p_n^{\mathbf{A}}(1, 0, 0) = 0,$$

a contradiction. \square

Remark 4. This theorem remains true without the assumption that \mathcal{V} is idempotent if (2) is changed to \mathbf{A} having a compatible order which is not an antichain.

Let Σ be an idempotent set of equations. Define the *order derivative* of Σ , denoted Σ^+ , to be the augmentation of Σ by additional equations as follows. If

$$\Sigma \models x \approx F(\mathbf{w})$$

where \mathbf{w} is a vector of not necessarily distinct variables and F is an operation symbol occurring in Σ , then Σ^+ contains all equations

$$x \approx F(\mathbf{w}')$$

where for each i , $w'_i = x$ or w_i .¹

Theorem 5. *Let Σ be a idempotent set of equations and \mathcal{V} a variety. Then*

- (1) *If \mathcal{V} realizes Σ and some iterated order derivative of Σ is inconsistent then \mathcal{V} is congruence n -permutable for some n .*
- (2) *If \mathcal{V} is congruence n -permutable for some n then \mathcal{V} realizes some Σ whose k^{th} iterated order derivative is inconsistent for some k .*

Proof. First we note that (1) holds for \mathcal{V} if and only if it holds for \mathcal{V}_Σ by a standard argument. Hence we may assume \mathcal{V} is idempotent. To see (1) assume \mathcal{V} realizes Σ and is not congruence n -permutable for any n . By Theorem 3 there is an $\mathbf{A} \in \mathcal{V}$ having a bounded compatible order.

Suppose Σ implies

$$(+) \quad x \approx F(\mathbf{u}, y, \mathbf{v})$$

Where \mathbf{u} and \mathbf{v} are vectors of not necessarily distinct variables. For each variable z_i occurring in \mathbf{u} or \mathbf{v} except x and y we substitute an element $c_i \in A$. For x we substitute an element $a \in A$. Let $\mathbf{u}(b)$ be \mathbf{u} with the above substitution and substituting b for y . $\mathbf{v}(b)$ is defined similarly. By (+)

$$\begin{aligned} a &= F(\mathbf{u}(0), 0, \mathbf{v}(0)) \\ &\leq F(\mathbf{u}(b), a, \mathbf{v}(b)) \\ &\leq F(\mathbf{u}(1), 1, \mathbf{v}(1)) = a \end{aligned}$$

showing \mathbf{A} satisfies $x \approx F(\mathbf{u}, x, \mathbf{v})$. Repeated applications of this argument show that \mathbf{A} is a model of Σ^+ . Hence Σ^+ is consistent and, by Theorem 3, does not imply n -permutability for any n . Repeating this argument we see that every order derivative of Σ is consistent.

For (2) suppose that \mathcal{V} is congruence n -permutable. Then \mathcal{V} realizes the Hagemann-Mitschke terms of Theorem 2(2). An easy inductive arguments shows that the k^{th} order derivative of Σ implies that $x \approx p_k(x, x, y)$. Thus the n^{th} order derivative implies $x \approx p_n(x, x, y) \approx y$. \square

Example 6. Recall that an algebra is *congruence regular* if whenever two congruences have a common block, they are equal. A variety is congruence regular if all of its members are. This property can be characterized by a Maltsev condition: a variety \mathcal{V} is congruence regular if and only if, for

¹This is stronger than the definition in an earlier version of this preprint.

some n , it realizes Σ , where Σ is:

$$\begin{aligned} g_i(x, x, z) &\approx z & 1 \leq i \leq n \\ x &\approx f_1(x, y, z, z, g_1(x, y, z)) \\ f_1(x, y, z, g_1(x, y, z), z) &\approx f_2(x, y, z, z, g_2(x, y, z)) \\ f_2(x, y, z, g_2(x, y, z), z) &\approx f_3(x, y, z, z, g_3(x, y, z)) \\ &\vdots \\ f_n(x, y, z, g_n(x, y, z), z) &\approx y \end{aligned}$$

[Give some references.] Clearly Σ^+ has the equation $g_i(z, x, z) \approx z$. Using this and substituting $z \mapsto x$ we see

$$f_i(x, y, x, x, g_i(x, y, x)) \approx f_i(x, y, x, g_i(x, y, x), x)$$

From this we see $\Sigma^+ \models x \approx y$. Thus Σ^+ is inconsistent and so congruence regular varieties are k permutable, for some k .

Of course Σ^+ inconsistent implies Σ' is. Thus congruence regular varieties are also congruence modular. Both of these results are unpublished results of J. Hagemann.

For an example showing that the converse of Theorem 5(1) is not true we can take Σ to be the equations defining the variety of idempotent quasi-groups. The operations symbols are \cdot , $/$, and \backslash . Besides the idempotent laws, Σ has the equations

$$\begin{aligned} x \cdot (x \backslash y) &\approx y & (x/y) \cdot y &\approx x \\ x \backslash (x \cdot y) &\approx y & (x \cdot y)/y &\approx x \end{aligned}$$

This variety is nontrivial: for $a \neq 0$ or 1 in a field, let $x \cdot y = ax + (1 - a)y$ and $x/y = y \backslash x = a^{-1}x + (1 - a^{-1})y$. It is also congruence permutable. For example

$$p(x, y, z) = (x/(y \backslash y)) \cdot (y \backslash z) \approx (x/y) \cdot (y \backslash z)$$

is a Maltsev term; see [2]. However, since the operation symbols are all binary it is easy to see that $\Sigma^+ = \Sigma$.

On the other hand, as in [1], when Σ is linear we do have a converse:

Theorem 7. *The following are equivalent for a set Σ of linear idempotent equations.*

- (1) *Some iterated order derivative is inconsistent.*
- (2) *Any variety that realizes Σ is congruence n -permutable for some n .*
- (3) *The variety \mathcal{V}_Σ axiomatized by Σ is congruence n -permutable for some n .*

Proof. (1) implies (2) follows from the last theorem and (2) implies (3) is clear. To see (3) implies (1) suppose (1) fails. So assume every iterated order derivative is consistent. Let Ω be the union of all the order derivatives. Then $\Omega^+ = \Omega$.

Let \mathbf{V} be the algebra on $\{0, 1\}$ such that for each operation symbol F occurring in Ω we have $F^{\mathbf{V}}(v_1, \dots, v_n) = 1$ if and only if

$$\Omega \models F(x_{v_1}, \dots, x_{v_n}) \approx x_1.$$

Kearnes, Kiss and Szendrei have shown in an unpublished manuscript (see [1]) that \mathbf{V} is a model of Ω . Using $\Omega^+ = \Omega$, it is easy to see that the operations of \mathbf{V} preserve the order on \mathbf{V} . Thus by Theorem 3(2) \mathcal{V}_Ω is not congruence n -permutable for any n . Since $\mathcal{V}_\Omega \subseteq \mathcal{V}_\Sigma$ it is also not congruence n -permutable for any n . \square

The following theorem was proved in [4] for locally finite varieties.

Theorem 8. *For a variety \mathcal{V} the following are equivalent.*

- (1) \mathcal{V} is congruence n -permutable for some n .
- (2) There is a linear idempotent set of equations Σ such that \mathcal{V} realizes Σ but the variety of distributive lattices \mathcal{D} does not.

Proof. The Hagemann-Mitschke terms witness that (1) implies (2).

For the other direction suppose \mathcal{V} is not n -permutable for any n and that \mathcal{V} realizes a linear idempotent set of equations Σ . By Theorem 7 every iterated order derivative of Σ is consistent. As in the proof of that theorem let Ω be the union of all order derivatives of Σ and let \mathbf{V} be the algebra on $\{0, 1\}$ given in that proof that models Ω . As before all of the operations are order preserving. Thus for the operation symbol F occurring in Ω , the operation $F^{\mathbf{V}}$ preserves order. Every nonconstant, order-preserving function on the two element lattice is a term function, so there is a lattice term that is equal to $F^{\mathbf{V}}$. Thus the variety \mathcal{D} of distributive lattices realizes Ω and hence Σ . \square

Modifying Corollary 5.3 in [1] we get the following corollary. [Add an explanation of how this works.]

Corollary 9. *The following problem is decidable: for a finite set Σ of idempotent linear equations determine if the realization of Σ in a variety implies congruence n -permutability for some n .*

2. SEMIDISTRIBUTIVITY

Define the *weak derivative* of Σ , denoted Σ^* , to be the augmentation of Σ by equations expressing F is independent of its i^{th} place if Σ implies

$$(*) \quad F(x, \dots, x, y, x, \dots, x) \approx x,$$

with y in the i^{th} place. This concept is both interesting and disappointing for the same reason: an analog of Theorem 4.2 of [1] and Theorem 5 of this paper hold but the analog of Theorem 5.2 of [1] and Theorem 7 do not. Despite this we will show that there is a recursive procedure to decide if a finite, idempotent, linear set of equations imply congruence semidistributivity (and congruence meet-semidistributivity).

Lemma 10. *Let Σ be a finite and idempotent set of equations. Then the following are equivalent.*

- (a) *If a variety \mathcal{V} realizes Σ , then \mathcal{V} is congruence meet semidistributive.*
- (b) *\mathcal{V}_Σ is congruence meet semidistributive.*
- (c) *Σ is not realized in any nontrivial variety of modules.*

This is just part of Theorem 8.1 of [6] from the point of view of Σ : (a) \Leftrightarrow (b) by a standard argument; Theorem 8.1 (10) \Rightarrow (1) gives (c) \Rightarrow (b); (a) \Rightarrow (c) is clear since no nontrivial variety of modules is congruence meet semidistributive.

Theorem 11. *Let Σ be a finite and idempotent set of equations. Then*

- (1) *If \mathcal{V} realizes Σ and some iterated weak derivative of Σ is inconsistent then \mathcal{V} is congruence semidistributive.*
- (2) *If \mathcal{V} is congruence semidistributive then \mathcal{V} realizes some Σ whose k^{th} iterated weak derivative is inconsistent for some k .*

Proof. To see (1) assume \mathcal{V} is not congruence semidistributive but realizes a set of equations Σ whose k^{th} iterated weak derivative is inconsistent. Since $\Sigma^* \subseteq \Sigma'$, if all derivatives of Σ are consistent then so are all weak derivatives. So we can assume some derivative is inconsistent and thus \mathcal{V} satisfies a nontrivial congruence identity. We are assuming \mathcal{V} is not congruence semidistributive, so by Theorem 8.14 (1) \Leftrightarrow (6) of [6], \mathcal{V} is not congruence meet semidistributive. By the lemma Σ must be realized in some nontrivial variety of modules \mathcal{M} . But if F is a module term and (*) holds it is easy to see that F is independent of its i^{th} place. Thus \mathcal{M} also satisfies Σ^* and all of iterated weak derivatives. In particular, all iterated weak derivatives are consistent.

For (2) suppose \mathcal{V} is congruence semidistributive. Then it realizes a variant in [10], call it Σ , of the Hobby-McKenzie-Kearnes-Kiss Maltsev condition for congruence semidistributivity, which says there are terms $d_i(x, y, z)$, $i = 0, \dots, k$ such that $d_0(x, y, z) \approx x$, $d_k(x, y, z) \approx z$ and for each i two of the following three hold:

- (1) $d_i(x, x, y) \approx d_{i+1}(x, x, y)$
- (2) $d_i(x, y, x) \approx d_{i+1}(x, y, x)$
- (3) $d_i(x, y, y) \approx d_{i+1}(x, y, y)$

Assume by induction on i that some iterated weak derivative of Σ implies $x \approx d_i(x, y, z)$. If the first two equations above hold then the next weak derivative implies that $d_{i+1}(x, y, z)$ is independent of its second and third variable and so $d_{i+1}(x, y, z) \approx x$. If, say, the first and third hold, then the next weak derivative implies that $d_{i+1}(x, y, z)$ is independent of its third variable and this together with the third equation implies the second equation. (So two weak derivatives were needed to go from i to $i+1$ in this case.) So some iterate of the weak derivative implies $x \approx d_k(x, y, z) \approx z$. \square

As mentioned above, the analogs of Theorem 5.2 of [1] and Theorem 7 do not hold. Here's an example showing that it may not be easy to get something that works: consider Σ to be the equations

$$\begin{aligned} p(x, y, y) &\approx p(y, y, x) \approx x \\ f(x, x, x, y) &\approx f(x, x, y, x) \approx f(x, y, x, x) \approx f(y, x, x, x) \\ g(x, x, y) &\approx g(x, y, x) \approx g(y, x, x) \\ f(x, x, x, x) &\approx g(x, x, x) \approx x \\ f(x, x, x, y) &\approx g(x, x, y) \end{aligned}$$

(f and g are weak nu terms; p is a Maltsev term.) These equations imply semidistributivity (actually distributivity since it has a Maltsev term). But without the last equation, they do not. This seems to indicate that a derivative condition for congruence semidistributivity could not just look at the function symbols individually. [Give credit to Kozik, *et al.*]

On the other hand there is a simple recursive procedure to decide if (the realization of) Σ implies congruence semidistributivity (or meet semidistributive) when Σ is linear (and idempotent): use Theorem 5.2 of [1] to test if Σ implies a nontrivial congruence identity (skip this if you are only testing for meet semidistributivity) and then check that Σ cannot be realized in a nontrivial variety of modules:

Theorem 12. *Given a finite set of linear equations Σ one can recursively decide if there is a nontrivial ring \mathbf{R} such that the variety of \mathbf{R} -modules realizes Σ .*

Proof. In [5], Hutchinson and Czedli give a recursive procedure to decide for which rings \mathbf{R} a given lattice identity holds in the congruence lattices of \mathbf{R} -modules. They analyze the Σ that comes from the lattice identity but their analysis works for any linear Σ . In a manner that is well known, one can translate Σ into a set of existentially quantified ring equations with integer coefficients such that \mathbf{R} -modules realize Σ if and only if there is a solution in \mathbf{R} of these ring equations. [Expand this.]

When Σ is linear this has a simple form: each equation is the sum of integer multiples of the ring variables equal to an integer. This can be put in matrix form:

$$AX = B$$

where A is an $m \times n$ matrix over \mathbb{Z} , B is a column vector over \mathbb{Z} , and X is a column vector of ring variables. Let D be the Smith Normal Form of A . So D is diagonal and $d_{i,i} \mid d_{i+1,i+1}$, $d_{i,i} > 0$ for $i = 1, \dots, r$ and $d_{i,i} = 0$ for $i > r$. The $d_{i,i}$ are the invariant factors. Also there are matrices $P \in \text{GL}(m, \mathbb{Z})$ and $Q \in \text{GL}(n, \mathbb{Z})$ with

$$D = PAQ.$$

Let $C = PB$ and $Y = Q^{-1}X$. Then $AX = B$ can be solved in \mathbf{R} if and only if $DY = C$ can.

If $c_j = 0$ for $j > r$ then $DY = C$ can be solved over \mathbb{Q} . If $c_j \neq 0$ for some $j > r$ then any ring satisfying $DY = C$ must have characteristic dividing c_j . So one can now test if $DY = C$ has a solution in $\mathbb{Z}/p\mathbb{Z}$ for each prime dividing c_j . \square

[Give more details above and maybe also describe the Czedli-Hutchinson-Frobenius invariants.]

The next corollary summarizes properties P such that it is known to be decidable, given a finite set Σ of linear idempotent equations, if every variety that realizes Σ satisfies P . The first two are from [1]. The proof is a straightforward modification of Corollary 5.3 in [1].

Corollary 13. *Each of the following problems is decidable: for a finite set Σ of idempotent, linear equations, determine for a variety if*

- (1) *the realization of Σ implies congruence modularity.*
- (2) *the realization of Σ implies a nontrivial congruence identity.*
- (3) *the realization of Σ implies congruence n -permutability, for some n .*
- (4) *the realization of Σ implies congruence meet-semidistributivity.*
- (5) *the realization of Σ implies congruence semidistributivity.*
- (6) *the realization of Σ implies congruence distributivity.*

[Matt, the example below is not part of the paper—it just gives a nice example of the use of the Dent-Kearnes-Szendrei method applied to our SD(Join) terms. The “your” in it refers to Keith and Agi.]

An Example. Let Σ_1 be

$$\begin{aligned} t(x, x, x, x, y) &\approx t(x, x, x, y, x) \approx t(x, x, y, x, x) \approx x \\ t(x, x, y, y, y) &\approx t(x, y, y, y, y) \approx t(y, x, y, y, y) \end{aligned}$$

and let Σ_2 be

$$\begin{aligned} s(x, x, x, x, y) &\approx s(x, x, x, y, x) \approx x \\ s(x, x, x, y, y) &\approx s(x, x, y, y, y) \approx s(x, y, x, y, y) \approx s(y, x, x, y, y) \end{aligned}$$

These come from the paper that Ross calls Matt *et al.*, *On Maltsev Conditions associated with omitting certain type of local structures*. Both s and t are Hobby-McKenzie terms and can't be realized in a nontrivial module, so imply congruence semidistributivity. Do they imply CD?

Σ'_1 is inconsistent so by Theorem 3.2 Σ_1 implies modularity and hence distributivity: the top equations imply Σ'_1 gives that t is independent of its last 3 variables. So

$$x \approx t(x, x, *, *, *) \approx t(x, y, *, *, *) \approx t(y, x, *, *, *)$$

Reversing x and y gives $y \approx t(x, y, *, *, *) \approx t(y, x, *, *, *)$, so $x \approx y$.

On the other hand Σ_2 does not imply distributivity. You can just construct the two-element algebras \mathbf{V} and \mathbf{V}' as in your proof of Theorem 5.1 and check that \mathbf{N}_5 is the congruence lattice of their product. (Naturally I used the UA Calculator to do this.)

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